

Dimension and natural parametrization for SLE curves

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Abstract

Some possible definitions for the natural parametrization of SLE paths are proposed in terms of various limits. One of the definitions is used to give a new proof that the Hausdorff dimension of SLE_κ paths is $1 + \frac{\kappa}{8}$ for $\kappa < 8$.

1 Introduction

A number of measures on paths or clusters on two-dimensional lattices arising from critical statistical mechanical models are believed to exhibit some kind of conformal invariance in the scaling limit. The Schramm-Loewner evolution (SLE) was created by Schramm [12] as a candidate for the scaling limit of discrete measures on paths arising in statistical physics. SLE is a continuous process which has the conformal invariance built in — in other words, it gives possible (and in some cases, the only possible) candidates for scaling limits assuming that these limits are conformally invariant.

To give an example, let us consider the case of self-avoiding walks (SAWs). We will not be very precise here; in fact, what we say here about SAWs is still only conjectural. Let $D \subset \mathbb{C}$ be a bounded domain, which for ease we will assume has a smooth boundary, and let z, w be distinct points on ∂D . Suppose that a lattice $\epsilon\mathbb{Z}^2$ is placed on D and let $\tilde{z}, \tilde{w} \in D$ be lattice points in $\epsilon\mathbb{Z}^2$ “closest” to z, w . A self-avoiding walk (SAW) ω from \tilde{z} to \tilde{w} is a sequence of distinct points

$$\tilde{z} = \omega_0, \omega_1, \dots, \omega_k = \tilde{w},$$

with $\omega_j \in \epsilon\mathbb{Z}^2 \cap D$ and $|\omega_j - \omega_{j-1}| = \epsilon$ for $1 \leq j \leq k$. We write $|\omega| = k$. For each $\beta > 0$, we can consider the measure on SAWs from \tilde{z} to \tilde{w} in D that gives measure $e^{-\beta|\omega|}$, to each such SAW. There is a critical β_0 , such that the partition function

$$\sum_{\omega: \tilde{z} \rightarrow \tilde{w}, \omega \subset D} e^{-\beta_0|\omega|}$$

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neither grows nor decays exponentially as a function of ϵ as $\epsilon \rightarrow 0$. It is believed that if we choose this β_0 , and normalize so that this is a probability measure, then there is a limiting measure on paths that is the scaling limit.

It is believed that the typical length of a SAW in the measure above is of order ϵ^{-d} where the exponent $d = 4/3$ can be considered the fractal dimension of the paths. For fixed ϵ , let us define the scaled function

$$\hat{\omega}(j\epsilon^d) = \omega_j, \quad j = 0, 1, \dots, |\omega|.$$

We can use linear interpolation to make this a continuous path $\hat{\omega} : [0, \epsilon^d |\omega|] \rightarrow \mathbb{C}$. Then it is conjectured that the following is true:

- As $\epsilon \rightarrow 0$, the above probability measure on paths converges to a probability measure $\mu_D^\#(z, w)$ supported on continuous curves $\gamma : [0, t_\gamma] \rightarrow \mathbb{C}$ with $\gamma(0) = z, \gamma(t_\gamma) = w, \gamma(0, t_\gamma) \subset D$.
- The probability measures $\mu_D^\#(z, w)$ are conformally invariant. To be more precise, suppose $F : D \rightarrow D'$ is a conformal transformation that extends to ∂D at least in neighborhoods of z and w . For each γ in D connecting z and w consider the paths $F \circ \gamma$ (we define this precisely below). Then this gives a measure we call $F \circ \mu_D^\#(z, w)$. The conformal invariance assumption is

$$F \circ \mu_D^\#(z, w) = \mu_{D'}^\#(F(z), F(w)).$$

Let us now define $F \circ \gamma$. The path $F \circ \gamma$ will traverse the points $F(\gamma(t))$ in order; the only question is how “quickly” does the curve traverse these points. If we look at how the scaling limit is defined, we can see that if $F(z) = rz$ for some $r > 0$, then the lattice spacing ϵ on D corresponds to lattice space $r\epsilon$ on $F(D)$ and hence we would expect the time to traverse $r\gamma$ should be r^d times the time to traverse γ . Using this as a guide locally, we say that the amount of time needed to traverse $F(\gamma[t_1, t_2])$ is

$$\int_{t_1}^{t_2} |F'(\gamma(s))|^d ds. \tag{1}$$

This tells us how to parametrize $F \circ \gamma$ and we include this as part of the definition of $F \circ \gamma$. This is analogous to the known conformal invariance of Brownian motion in \mathbb{C} where the time parametrization must be defined as in (1) with $d = 2$.

We could weaken our conjecture and only consider γ and $F \circ \gamma$ as being defined only up to reparametrization. In this case, one can show that the only candidate for the scaling limit is (chordal) Schramm-Loewner evolution (SLE_κ) for some $\kappa > 0$; we define it more precisely later but we give the basics here. If there is to be a family of probability measures $\mu_D^\#(z, w)$ (now being considered modulo reparametrization) for simply connected D , then we only need to define $\mu_{\mathbb{H}}^\#(0, \infty)$. If we impose one further “domain Markov” property (which must be satisfied by a scaling limit of SAW), then there is a one-parameter family of probability

measures $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ from which the measure must come. In the case of the self-avoiding walk, another property called the “restriction property” tells use which particular choice in this family is the scaling limit [8, 9], but other choices arise as scaling limits of other models.

Schramm’s construction starts by giving a different parametrization to the curve γ in terms of a capacity in \mathbb{H} from infinity. When the curve γ is given this parametrization and g_t denotes a conformal map from the slit domain $\mathbb{H} \setminus \gamma[0, t]$ to \mathbb{H} , then $g_t(z)$ as a function satisfies a differential equation which goes back to Loewner. This parametrization can also be defined on the discrete level, and it has been shown that for some models (e.g., loop-erased random walk [7], harmonic explorer [13]), the discrete model with capacity parametrization converges to SLE with capacity parametrization.

In this paper we consider two closely related questions for SLE_κ :

- Given $\gamma(t)$ parametrized by capacity, can one recover the “natural parametrization”?
- How do we compute (rigorously) the Hausdorff dimension of the path $\gamma[0, t]$?

It is known that for $\kappa \geq 8$, the paths of SLE_κ fill the plane. Let us restrict our discussion to $\kappa < 8$. For these κ , it is known that the Hausdorff dimension of $\gamma[0, t]$ is given by

$$d = d_\kappa = 1 + \frac{\kappa}{8}.$$

In the case $\kappa = 8/3$, this was first proved by Schramm, Werner, and the author (see [6] and references therein) using the relationship between $SLE_{8/3}$ and the outer boundary of planar Brownian motion. The upper bound for general κ was first established by Rohde and Schramm [11] by calculating the expectation of a derivative; we give a form of this argument in Section 6. The lower bound is much harder to establish. This was done by Beffara [1] using very intricate estimates.

Establishing the lower bound is closely related to the question of finding the natural parametrization. Suppose that we can find

$$\tau : [0, \infty) \rightarrow [0, \infty),$$

such that $\eta(t) = \gamma(\tau(t))$ gives SLE_κ in the natural parametrization. Then we expect that η induces a d -dimensional measure on the path γ in the sense that for all $\alpha < d$

$$\int_0^1 \int_0^1 \frac{ds dt}{|\eta(s) - \eta(t)|^\alpha} < \infty. \quad (2)$$

Frostman’s lemma (see Section 3.2) tells us that the above condition implies that the Hausdorff dimension of the path is at least d . A weaker version of (2) is sufficient to establish the lower bound on Hausdorff dimension: it suffices to find a (perhaps random) subset J of $[0, 1]$ such that

$$\int_0^1 1_J(t) dt > 0, \quad \int_0^1 \int_0^1 \frac{1_J(s) 1_J(t) ds dt}{|\eta(s) - \eta(t)|^\alpha} < \infty. \quad (3)$$

In the next section, we will discuss a number of candidates for the natural parametrization. We conjecture that they are all equivalent up to multiplicative constant. They are all described in terms of limits that are hard to establish. Although we do not prove the limits exist, we do prove a kind of tightness result for one of the definitions that allows us to take a subsequential limit and construct a Frostman measure. This gives a new proof of the lower bound for the Hausdorff dimension of SLE_κ paths that was first proved by Beffara. In [10] we will establish the existence of the natural parametrization at least for a range of κ including $\kappa = 8/3$.

The majority of this paper is the proof of the lower bound for the Hausdorff dimension combined with the derivative estimates needed to establish a second moment bound. To give an idea, we would like to compare our construction of a Frostman measure to that in [1]. Let γ be an SLE_κ curve with $\kappa = 2/a < 8$. In [1], the starting point is to show that for fixed $z \in \mathbb{H}$, as $\delta \rightarrow 0+$,

$$\mathbb{P}\{\text{dist}[z, \gamma(0, \infty)] \leq \delta\} \asymp G(z) \delta^{2-d}, \quad (4)$$

and to define a measure on $\gamma(0, 1]$ to be δ^{d-2} times area restricted to the set $\{z \in \mathbb{H} : \text{dist}[z, \gamma(0, 1)] \leq \delta\}$. Here $G(z)$ denotes the “Green’s function” for chordal SLE_κ in \mathbb{H} ,

$$G(y(x+i)) = y^{d-2} (x^2 + 1)^{\frac{1}{2}-2a}.$$

The estimate (4) can be used to show that $\mathbb{E}[|\mu_\delta|] \asymp 1$ as $\delta \rightarrow 0+$. Here $|\cdot|$ denotes total mass. A much harder argument establishes a two-point estimate

$$\mathbb{P}\{\text{dist}[z, \gamma(0, \infty)] \leq \delta, \text{dist}[w, \gamma(0, \infty)] \leq \delta\} \leq c \left(\frac{\delta}{|z-w|} \right)^{2-d} \delta^{2-d}, \quad |z-w| \geq \delta. \quad (5)$$

Once (5) is established, standard arguments, see, e.g. [6], can be used to construct a Frostman measure. Unfortunately, (5) is not at all easy to prove. The problem is that the estimates used to prove (4) work well for a fixed z but do not handle two points z, w as well.

A somewhat different approach is taken in this paper. We sketch the idea here. Suppose that g_t are the conformal maps of SLE with driving function V_t where V_t is a standard Brownian motion. Let $\hat{f}_t(z) = g_t^{-1}(z + V_t)$. The Frostman measure is defined (approximately) to be the limit of the measures μ_n where μ_n gives measure

$$n^{-d/2} |\hat{f}'_{\frac{j-1}{n}}(i/\sqrt{n})|^d$$

to $\gamma[\frac{j-1}{n}, \frac{j}{n}]$. This approach also requires giving a second moment bound as well as showing that the curve γ is sufficiently “spread out” so that the limit measure satisfies (2). We, in fact, consider a submeasure by considering only certain good times and establishing a bound as in (3). This method avoids complications for $\kappa \geq 4$ where SLE curves have double points and near double points. Roughly speaking, we consider a subset J of times for which $|\eta(s) - \eta(t)| \approx |s - t|^{1/d}$ for $s, t \in J$.

The main hurdle in this paper, which is also important in [10], is to estimate expectations of the form

$$\mathbb{E} \left[|\hat{f}'_s(z)|^d |\hat{f}'_{s+t}(w)|^d \right], \quad z, w \in \mathbb{H}.$$

This expectation can be written as

$$\mathbb{E} \left[|h'_{s+t}(w)|^d |\tilde{h}'_s(z)|^d \right],$$

where h_{s+t}, \tilde{h}_s are conformal maps coming from the reverse Loewner flow (see Section 3.1 for definitions). Moreover,

$$h'_{s+t}(w) = h'_t(w) \tilde{h}'_s(Z_t(w)),$$

where h_t is independent of \tilde{h}_s and $Z_t(w) = h_t(w) - U_t$ where U_t is the driving function for the reverse flow. Hence, the expectation can be written as

$$\mathbb{E} \left[|h'_t(w)|^d |\tilde{h}'_s(Z_t(w))|^d |\tilde{h}'_s(z)|^d \right].$$

Although the maps h_t and \tilde{h}_s are independent, the random variable $|\tilde{h}'_s(Z_t(w))|$ is not independent of $|h'_t(w)|$ since $Z_t(w)$ depends on h_t . In order to estimate this expectation one needs to consider the distribution of $Z_t(w)$ when one weights paths proportionally to $|h'_t(w)|^d$. This is the main topic of Section 5 where the Girsanov theorem is combined with a particular martingale to study this quantity.

Here is the basic outline of the paper. The general discussion of natural parametrization is in Section 2. The next section concerns preliminary results: the reverse-time Loewner flow, a general result about computing Hausdorff dimension of random curves, and a simple distortion result for conformal maps. The proof of the lower bound for Hausdorff dimension is done in Section 4. Here, the approximations to the Frostman measure is defined precisely and it is shown how to derive the correlation estimates needed to establish the limit as a Frostman measure. The results in this section rely on one lemma about the reverse Loewner flow at one point. This lemma, as well as a number of related results, are discussed in Section 5. For completeness we finish with two sections quickly redoing results from [11]. We prove the upper bound for the Hausdorff dimension and we show the existence of the curve for $\kappa < 8$. This latter result is used in the proof of our main theorem in Section 4 (although we could have avoided using it) so we decided to include it here.

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2 Natural parametrization

2.1 Schramm-Loewner evolution (*SLE*)

We start by giving a quick review of the definition of the Schramm-Loewner evolution; see [6], especially Chapters 6 and 7, for more details. We will discuss only chordal *SLE* in this paper, and we will call it just *SLE*.

Suppose that $\gamma : (0, \infty) \rightarrow \mathbb{H} = \{x + iy : y > 0\}$ is a non-crossing curve with $\gamma(0+) \in \mathbb{R}$ and $\gamma(t) \rightarrow \infty$ as $t \rightarrow \infty$. Let H_t be the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$. Using the Riemann mapping theorem, one can see that there is a unique conformal transformation

$$g_t : H_t \longrightarrow \mathbb{H}$$

satisfying $g_t(z) - z \rightarrow 0$ as $z \rightarrow \infty$. It has an expansion at infinity

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}).$$

The coefficient $a(t)$ equals $\text{hcap}(\gamma(0, t])$ where $\text{hcap}(A)$ denotes the \mathbb{H} -capacity from infinity of a bounded set A . There are a number of ways of defining hcap , e.g.,

$$\text{hcap}(A) = \lim_{y \rightarrow \infty} y \mathbb{E}^{iy}[\text{Im}(B_\tau)],$$

where B is a complex Brownian motion and $\tau = \inf\{t : B_t \in \mathbb{R} \cup A\}$.

Definition The *Schramm-Loewner evolution*, SLE_κ , (from 0 to infinity in \mathbb{H}) is the random curve $\gamma(t)$ such that g_t satisfies

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \quad (6)$$

where $a = 2/\kappa$ and $U_t = -B_t$ is a standard Brownian motion.

Showing that the conformal maps g_t are well defined is easy. It is not as obvious that the curve γ exists. In Section 7 we give a version of the argument first given by Rohde and Schramm [11] that the curve exists for $\kappa \neq 8$. The argument actually shows more: there is a $\theta = \theta(\kappa) > 0$ such that with probability one for all $0 < T_1 < T_2 < \infty, 0 < y \leq 1$ there is a (random) C such that for that for all $T_1 \leq s < t \leq T_2$,

$$|\gamma(s) - \hat{f}_s(yi)| \leq Cy^\theta, \quad |\gamma(s) - \gamma(t)| \leq C(t - s)^\theta. \quad (7)$$

Although the argument here does not differ substantially from that in [11], we give it because we will use this result and it follows quickly from the derivative exponents that we derive.

Remark We have defined chordal SLE_κ so that it is *parametrized by capacity* with

$$\text{hcap}(\gamma(0, t]) = at.$$

It is more often defined with the capacity parametrization chosen so that $\text{hcap}(\gamma[0, t]) = 2t$. In this case we need to choose $U_t = -\sqrt{\kappa} B_t$. We will choose the parametrization in (6), but this is only for our convenience. Under our parametrization, if $z \in \mathbb{H} \setminus \{0\}$, then $Z_t = Z_t(z) := g_t(z) - U_t$ satisfies the Bessel equation

$$dZ_t = \frac{a}{Z_t} dt + dB_t.$$

We let

$$f_t = g_t^{-1}, \quad \hat{f}_t(z) = f_t(z + U_t).$$

Define $g_{t,s}$ by $g_{t+s} = g_{t,s} \circ g_s$ and note that for fixed s , $g_{t,s}$ is the solution of the initial value problem

$$\partial_t g_{t,s}(z) = \frac{a}{g_{t,s}(z) - U_{s+t}}, \quad g_{0,s}(z) = z. \quad (8)$$

We also write $f_{t,s} = g_{t,s}^{-1}$ and note that

$$g_t = g_{t,0}, \quad f_t = f_{t,0}, \quad f_{t+s} = f_s \circ f_{t,s}, \quad f_{t,s} = g_s \circ f_{t+s}. \quad (9)$$

We will make strong use of the following well known scaling relation [6, Proposition 6.5].

Lemma 2.1 (Scaling). *If $r > 0$, then the distribution of $g_{tr^2}(rz)/r$ is the same as that of $g_t(z)$; in particular, $g'_{tr^2}(rz)$ has the same distribution as $g'_t(z)$.*

For $\kappa < 8$, we let

$$d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a}. \quad (10)$$

We choose this notation because this is the Hausdorff dimension. However, we do not assume this result in this paper, and so for now this is only a choice of notation.

2.2 Candidates

Here we let $\kappa < 8$ and give a number of possible ways to obtain the natural parametrization. We expect that they all give the same value up to multiplicative constant. In each case we will define approximate parametrizations $\tau_n(t)$ and then the parametrization τ is given by

$$\tau(t) = \lim_{n \rightarrow \infty} \tau_n(t).$$

We leave open the question of what kind of limit can be taken here.

Any candidate for the natural parametrization should satisfy the appropriate scaling relationship. In particular if $\gamma(t)$ is an SLE_κ curve, parametrized so that $\text{hcap}[\gamma(0, t]] = at$, then $\tilde{\gamma}(t) = r\gamma(t)$ is an SLE_κ curve parametrized so that $\text{hcap}[\tilde{\gamma}(0, t]] = r^2 at$. If it takes time $\tau(t)$ to traverse $\gamma(0, t]$ in the natural parametrization, then it should take time $r^d \tau(t)$ to traverse $\tilde{\gamma}(0, t]$ in the natural parametrization. In particular, it takes time $O(R^d)$ in the natural parametrization to go distance R .

2.2.1 Minkowski measure

Let

$$\mathcal{D}_{t,\epsilon} = \{z \in \mathbb{H} : \text{dist}(z, \gamma(0, t]) \leq \epsilon\},$$

$$\tau_n(t) = n^{2-d} \text{area}(\mathcal{D}_{t,1/n}).$$

We call the limit, if it exists, the Minkowski measure of $\gamma(0, t]$. This terminology is somewhat misleading because this is not a measure. It is not too difficult to show that as $\epsilon \rightarrow 0+$,

$$\mathbb{P}\{z \in \mathcal{D}_{\infty, \epsilon}\} \asymp G(z) \epsilon^{2-d}, \quad (11)$$

where, as before, $G(y(x+i)) = y^{d-2}(x^2+1)^{\frac{1-4a}{2}}$. This shows that $\tau_n(t)$ has been scaled so its expectation stays bounded away from 0 and infinity. It is significantly more difficult to derive the second moment bound,

$$\mathbb{P}\{z, w \in \mathcal{D}_{\infty, \epsilon}\} \leq c G(z) G(w) \epsilon^{2(2-d)} |w - z|^{d-2}. \quad (12)$$

With this bound, one can use this definition to prove the lower bound on the Hausdorff dimension. This is the strategy used by Beffara and the hard work comes in proving (12). Even with the second moment bound, it is not known how to prove that the limit defining $\tau(t)$ exists.

There is a variant of the Minkowski measure that could be called the conformal Minkowski measure. Let g_t be the conformal maps as above. If $t < T_z$, let

$$\Upsilon_t(z) = \frac{\text{Im}[g_t(z)]}{|g'_t(z)|}.$$

A simple calculation shows that $\Upsilon_t(z)$ decreases in t and hence we can define

$$\Upsilon_t(z) = \Upsilon_{T_z-}(z), \quad t \geq T_z.$$

Similarly, $\Upsilon(z) = \Upsilon_{\infty}(z)$ is well defined. The Koebe 1/4-Theorem can be used to show that $\Upsilon_t(z) \asymp \text{dist}[z, \gamma(0, t] \cup \mathbb{R}]$; in fact, each side is bounded above by four times the other side. To prove (11) one can show that there is a c_* such that

$$\mathbb{P}\{\Upsilon(z) \leq \epsilon\} \sim c_* G(z) \epsilon^{2-d}, \quad \epsilon \rightarrow 0+.$$

This was first established in [6] building on the argument in [11]. We give another proof of this in Section 6. The conformal Minkowski measure is defined as in the previous paragraph replacing $\mathcal{D}_{t, \epsilon}$ with

$$\mathcal{D}_{t, \epsilon}^* = \{z \in \mathbb{H} : \Upsilon_t(z) \leq \epsilon\}.$$

It is possible that this limit will be easier to establish. Assuming the limit exists, we can see that the expected amount of time (using the natural parametrization) that $\gamma(0, \infty)$ spends in a bounded domain D should be given (up to multiplicative constant) by

$$\int_D G(z) dA(z), \quad (13)$$

where A denotes area. This observation is the starting point for the construction of the natural parametrization in [10]. However, we will not use either version of the Minkowski measure in this paper.

2.2.2 d -variation

Let

$$\tau_n(t) = \sum_{k=1}^{\lfloor tn \rfloor} \left| \gamma\left(\frac{k}{n}\right) - \gamma\left(\frac{k-1}{n}\right) \right|^d.$$

More generally, we can consider

$$\tau_n(t) = \sum_{t_{j-1,n} < t} |\gamma(t_{j,n}) - \gamma(t_{j-1,n})|^d,$$

where $t_{0,n} < t_{1,n} < t_{2,n} < \infty$ is a partition, depending on n , whose mesh goes to zero as $n \rightarrow \infty$. One expects that for a wide class of partitions this limit exists and is independent of the choice of partitions. In the case $\kappa = 8/3$, a version of this was studied numerically by Kennedy [5].

2.2.3 Derivatives of inverse map

We now describe the definition that will be used in this paper. We start with some heuristics. Suppose $\tau(t)$ were the natural parametrization. Since $\tau(1) < \infty$, we would expect that the average value of

$$\Delta_n \tau(j) := \tau\left(\frac{j+1}{n}\right) - \tau\left(\frac{j}{n}\right)$$

would be of order $1/n$. Consider

$$\gamma^{(j/n)}\left[0, \frac{1}{n}\right] = g_{j/n}\left(\gamma\left[\frac{j}{n}, \frac{j+1}{n}\right]\right).$$

Since the hcap of this set is a/n , we expect that the diameter of the set is of order $1/\sqrt{n}$. Using the scaling properties, we guess that the time needed to traverse $\gamma^{(j/n)}[0, \frac{1}{n}]$ in the natural parametrization is of order $n^{-d/2}$. Using the scaling properties again, we guess that

$$\Delta_n \tau(j) \approx n^{-d/2} |\hat{f}'_{j/n}(i/\sqrt{n})|^d.$$

This leads us to define

$$\tau_n(t) = \sum_{k=1}^{\lfloor tn \rfloor} n^{-d/2} |\hat{f}'_{k/n}(i/\sqrt{n})|^d. \tag{14}$$

This is the form we will use.

More generally, we could let

$$\tau_n(t) = \sum_{k=1}^{\lfloor tn \rfloor} n^{-d/2} \int_{\mathbb{H}} |\hat{f}'_{k/n}(z/\sqrt{n})|^d \nu(dz),$$

where ν is a finite measure on \mathbb{H} . This approach used in [10] to construct the natural parametrization starts with (13) but ends up with a version of this for a particular measure ν .

2.3 Lower bound on Hausdorff dimension

In this paper, we will use the ideas from Section 2.2.3 to prove that for $\kappa < 8$, the Hausdorff dimension of the paths is $d = 1 + \frac{\kappa}{8}$. We will focus on the proof of the lower bound which is the hard direction. (For completeness sake we sketch a proof of the upper bound in Section 6.) Since Hausdorff dimension is preserved under conformal maps, it is easy to use the independence of the increments of Brownian motion to conclude that there is a d_* such that with probability one $\dim_h[\gamma[t_1, t_2]] = d_*$ for all $t_1 < t_2$. Using this and the upper bound, we can see that it suffices to prove that for all $\alpha < d$,

$$\mathbb{P}\{\dim_h(\gamma[1, 2]) \geq \alpha\} > 0.$$

Since $\mathbb{E}[\tau_n(t)] \asymp 1$, (14) suggests the relation

$$\mathbb{E}[|\hat{f}'_1(i/\sqrt{n})|^d] = \mathbb{E}[|\hat{f}'_n(i)|^d] \asymp n^{\frac{d}{2}-1}. \quad (15)$$

(The first equality holds by scaling.) This was proved in [11]. We give another proof here that derives additional information. Let

$$\beta = d - \frac{3}{2} = \frac{1 - 2a}{4a} = \frac{\kappa - 4}{8},$$

$$\xi = d(d - 2) + 1 = \beta d + 1 - \frac{d}{2} = \frac{1}{16a^2} = \frac{\kappa^2}{64}.$$

Note that $\beta < 1/2$ and $0 < \xi < 1$. In our proof of (15) one sees that $\mathbb{E}[|\hat{f}'_t(i)|^d]$ is not of the same order of magnitude as $\mathbb{E}[|\hat{f}'_t(i)|^d]$. Roughly speaking, the expectation of $|\hat{f}'_t(i)|^d$ is supported on the event that $|\hat{f}'_t(i)| \approx t^\beta$. In order to make this precise, it will be convenient to introduce some terminology.

Definition A function $\phi : [0, \infty) \rightarrow (0, \infty)$ is a *subpower function* if it is increasing, continuous, and

$$\lim_{x \rightarrow \infty} \frac{\log \phi(x)}{\log x} = 0,$$

i.e., ϕ grows slower than x^q for all $q > 0$.

The class of subpower functions is closed under addition and multiplication. In Theorem 5.1, it is proved that there is a subpower function ϕ such

$$\mathbb{E}[|\hat{f}'_t(i)|^d] \asymp \mathbb{E}\left[|\hat{f}'_t(i)|^d; t^\beta \phi(t)^{-1} \leq |\hat{f}'_t(i)| \leq t^\beta \phi(t)\right] \asymp t^{\frac{d}{2}-1},$$

which implies that

$$\mathbb{P}\left\{t^\beta \phi(t)^{-1} \leq |\hat{f}'_t(i)| \leq t^\beta \phi(t)\right\} \approx t^{\frac{d}{2}-1-\beta d} = t^{-\xi}. \quad (16)$$

The basic idea underlying our construction of a Frostman measure on the path is to replace $|\hat{f}'_{j/n}(i/\sqrt{n})|^d$ in (14) with $|\hat{f}'_{j/n}(i/\sqrt{n})|^d 1_{E_{j,n}}$ where $E_{j,n}$ is an event, measurable with respect to U_t , $0 \leq t \leq j/n$, which roughly corresponds to (the scaled version of) the event in (16).

3 Some preliminaries

3.1 Reverse time

It is known [11, 3] that estimates for \hat{f}_t' are often more easily derived by considering the reverse (time) Loewner flow. In this subsection, we review the facts about the Loewner equation in reverse time that we will need. Suppose that g_t is the solution to the Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z. \quad (17)$$

Here V_t can be any continuous function, but we will be interested in the case where V_t is a standard Brownian motion.

For fixed $T > 0$, let $F_t^{(T)}, 0 \leq t \leq T$, denote the solution to the time-reversed Loewner equation

$$\partial_t F_t^{(T)}(z) = -\frac{a}{F_t^{(T)}(z) - V_{T-t}} = \frac{a}{V_{T-t} - F_t^{(T)}(z)}, \quad F_0^{(T)}(z) = z. \quad (18)$$

Note that

$$F_{s+T}^{(S+T)}(z) = F_s^{(S)}(F_T^{(S+T)}(z)), \quad 0 \leq s \leq S.$$

Lemma 3.1. *If $t \leq T$, then $F_t^{(T)} = f_{t,T-t}$. In particular, $F_T^{(T)} = f_T$.*

Proof. Fix T , and let $u_t = F_{T-t}^{(T)}$. Then (18) implies that u_t satisfies

$$\dot{u}_t(z) = \frac{a}{u_t(z) - V_t}, \quad u_T(z) = z.$$

By comparison with (17), we can see that $u_t(z) = g_t(f_T(z))$, and we have already noted in (9) that $g_t \circ f_T = f_{t,T-t}$. \square

We will be using the reverse-time flow, to study the behavior of \hat{f} at one or two times. We leave the simple derivation of the next lemma from the previous lemma to the reader. A primary purpose of stating this lemma now is to set the notation for future sections.

Lemma 3.2. *Suppose $S, T > 0$ and $V : [0, S+T] \rightarrow \mathbb{R}$ is a continuous function. Suppose $g_t, 0 \leq t \leq S+T$ is the solution to (17). As before, let $f_t = g_t^{-1}$ and $\hat{f}(z) = f_t(z + V_t)$. Let*

$$U_t = V_{S+T-t} - V_{S+T}, \quad 0 \leq t \leq S+T,$$

$$\tilde{U}_t = V_{S-t} - V_S = U_{T+t} - U_T, \quad 0 \leq t \leq S.$$

and let $h_t, 0 \leq t \leq S+T$, $\tilde{h}_t, 0 \leq t \leq S$, be the solutions to the reverse-time Loewner equations

$$\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z,$$

$$\partial_t \tilde{h}_t(z) = \frac{a}{\tilde{U}_t - \tilde{h}_t(z)}, \quad \tilde{h}_0(z) = z.$$

Then

$$\begin{aligned}\hat{f}_S(z) &= \tilde{h}_S(z) - \tilde{U}_S, & \hat{f}_{S+T}(z) &= h_{S+T}(z) - U_{S+T}, \\ h_{S+T}(z) &= \tilde{h}_S(h_T(z) - U_T) + U_T.\end{aligned}$$

In particular,

$$\hat{f}'_S(w) \hat{f}'_{S+T}(z) = h'_T(z) \tilde{h}'_S(h_T(z) - U_T) \tilde{h}'_S(w).$$

If $S > T$, we will also need to consider \hat{f}_{S-T} . We will use $\hat{h}_t, \bar{U}_t, 0 \leq t \leq S - T$ for the corresponding quantities.

Remark If V_t is a Brownian motion starting at the origin, then U_t, \tilde{U}_t are standard Brownian motions starting at the origin. Moreover $\{U_t : 0 \leq t \leq T\}$ and $\{\tilde{U}_t : 0 \leq t \leq S\}$ are independent.

3.2 Hausdorff dimension

The main tool for proving lower bounds for Hausdorff dimension is Frostman's lemma (see [2, Theorem 4.13]), a version of which we recall here: if $A \subset \mathbb{R}^m$ is compact and μ is a Borel measure supported on A with $0 < \mu(A) < \infty$ and

$$\mathcal{E}_\alpha(\mu) := \int \int \frac{\mu(dx) \mu(dy)}{|x - y|^\alpha} < \infty,$$

then the Hausdorff- α measure of A is infinite. In particular, $\dim_h(A) \geq \alpha$. The next lemma is similar to many that have appeared before (see [6, A.3]), but the exact formulation is what is needed here.

Lemma 3.3. *Suppose $\eta : [0, 1] \rightarrow \mathbb{R}^m$ is a random curve and*

$$\{F(j, n) : n = 1, 2, \dots, j = 1, 2, \dots, n\}$$

are nonnegative random variables all defined on the same probability space. Suppose that there exist $0 < \delta < \delta' < 1, 0 < C_1, C_2, C_3, C_4 < \infty, 0 < \alpha < m$ such that the following hold for $n = 1, 2, \dots$ and $1 \leq j \leq k \leq n$:

$$C_1 \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}[F(j, n)] \leq C_2, \tag{19}$$

$$\mathbb{E}[F(j, n)F(k, n)] \leq C_3 \left(\frac{n}{k - j + 1} \right)^\delta, \tag{20}$$

and

$$\left| \eta\left(\frac{j}{n}\right) - \eta\left(\frac{k}{n}\right) \right|^\alpha \geq C_4 \left(\frac{k - j}{n} \right)^{1-\delta'} 1\{F(j, n)F(k, n) > 0\}. \tag{21}$$

Then

$$\mathbb{P} \{ \dim_h(\eta[0, 1]) \geq \alpha \} > 0.$$

Remark The proof constructs a measure supported on the curve. The n th approximation is a sum of measures $\mu_{j,n}$ which are multiples of Lebesgue measure on small discs centered at $\eta(j/n)$. The multiple at $\eta(j/n)$ is chosen so that the total mass $\mu_{j,n}$ is $F(j,n)/n$. In particular, if $F(j,n) = 0$, $\mu_{j,n}$ is the zero measure. To apply Frostman's lemma, we need to show that the limiting measure is sufficiently spread out and (21) gives the necessary assumption. Note the assumption only requires the inequality to hold when $F(j,n)F(k,n) > 0$. The assumption implies that if $j < k$ and $\eta(j/n) = \eta(k/n)$ (or are very close), then at most one of $\mu_{j,n}$ and $\mu_{k,n}$ is nonzero.

Proof. We fix $\epsilon > 0$ such that $\alpha \leq m - \epsilon$ and $\epsilon \leq \delta \leq \delta' - \epsilon \leq 1 - 2\epsilon$. Constants in this proof depend on m and ϵ . Note that (20) and (21) combine to give

$$\mathbb{E} \left[\frac{F(j,n) F(k,n)}{|\eta(j/n) - \eta(k/n)|^\alpha} \right] \leq \frac{C_3}{C_4} \left(\frac{n}{k-j} \right)^{1-(\delta'-\delta)}, \quad j < k. \quad (22)$$

Let $\mu_{j,n}$ denote the (random) measure which is a multiple of Lebesgue measure on the disk of radius $n^{-(1-\delta)/\alpha}$ about $\eta(j/n)$ where the multiple is chosen so that $|\mu_{j,n}| = n^{-1} F(j,n)$. Here $|\cdot|$ denotes total mass. Let $\nu_n = \sum_{j=1}^n \mu_{j,n}$. From (19), we see that

$$\mathbb{E}[|\nu_n|] \geq C_1,$$

and from (20) we see that

$$\mathbb{E}[|\nu_n|^2] = \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n \mathbb{E}[F(j,n) F(k,n)] \leq \frac{C_3}{n^2} \sum_{j=1}^n \sum_{k=1}^n \left(\frac{n}{|k-j|+1} \right)^\delta \leq c C_3,$$

We will now show that

$$\mathbb{E}[\mathcal{E}_\alpha(\nu_n)] = \sum_{j=1}^n \sum_{k=1}^n \mathbb{E} \left[\int \int \frac{\mu_{j,n}(dx) \mu_{k,n}(dy)}{|x-y|^\alpha} \right] \leq c \frac{C_3}{C_4 \wedge 1}, \quad (23)$$

using the easy estimate

$$\int_{|x-x_0| \leq r} \int_{|y-y_0| \leq r} \frac{d^m x d^m y}{|x-y|^\alpha} \leq c r^{2m} \min\{r^{-\alpha}, |x_0 - y_0|^{-\alpha}\}.$$

To estimate the terms with $j = k$, note that (20) gives

$$\mathbb{E} \left[\int \int \frac{\mu_{j,n}(dx) \mu_{j,n}(dy)}{|x-y|^\alpha} \right] \leq c \mathbb{E} \left[\frac{F(j,n)^2 n^{1-\delta}}{n^2} \right] \leq c C_3 n^{-1},$$

and hence

$$\sum_{j=1}^n \mathbb{E} \left[\int \int \frac{\mu_{j,n}(dx) \mu_{j,n}(dy)}{|x-y|^\alpha} \right] \leq c C_3.$$

For $j < k$, we use the estimate

$$\int \int \frac{\mu_{j,n}(dx) \mu_{k,n}(dy)}{|x - y|^\alpha} \leq c \frac{F(j, n) F(k, n)}{n^2 |\eta(j/n) - \eta(k/n)|^\alpha}.$$

Combining this with (22) gives,

$$\sum_{1 \leq j < k \leq n} \mathbb{E} \left[\int \int \frac{\mu_{j,n}(dx) \mu_{k,n}(dy)}{|x - y|^\alpha} \right] \leq c \frac{C_3}{C_4 n^2} \sum_{1 \leq j < k \leq n} \left(\frac{n}{k - j} \right)^{1 - (\delta' - \delta)} \leq c \frac{C_3}{C_4}.$$

This gives (23).

Standard arguments show that there is a $p > 0$, such that with probability at least p ,

$$|\nu_n| \geq p, \quad \mathcal{E}_\alpha(\nu_n) \leq 1/p \text{ for infinitely many } n.$$

On this event we can take a subsequential limit ν with $|\nu| \geq p$ and $\mathcal{E}_\alpha(\nu) \leq 1/p$. Since ν must be supported on $\eta[0, 1]$, the conclusion follows from Frostman's lemma. \square

Remark In the proof we chose $\mu_{j,n}$ to a multiple of Lebesgue measure on a small disk centered at $\eta(j/n)$ rather than choosing it to be a point mass at $\eta(j/n)$. We needed to do this in order to establish (23). If we did not spread out the measure a little bit, the terms in the sum with $j = k$ would be infinite.

Corollary 3.4. *Suppose $\eta : [0, 1] \rightarrow \mathbb{R}^m$ is a random curve and*

$$\{F(j, n) : n = 1, 2, \dots, j = 1, 2, \dots, n\}$$

are nonnegative random variables all defined on the same probability space. Suppose $1 < d \leq m$, and there exist a subpower function ψ , $0 < \xi < 1$, and $c < \infty$ such that the following holds for $n = 1, 2, \dots$, and $1 \leq j \leq k \leq n$.

$$c^{-1} \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}[F(j, n)] \leq c. \quad (24)$$

$$\mathbb{E}[F(j, n)F(k, n)] \leq \left(\frac{n}{k - j + 1} \right)^\xi \psi \left(\frac{n}{k - j + 1} \right), \quad (25)$$

and

$$\left| \eta \left(\frac{j}{n} \right) - \eta \left(\frac{k}{n} \right) \right|^d \geq \left(\frac{k - j}{n} \right)^{1 - \xi} \psi \left(\frac{n}{|j - k| + 1} \right)^{-1} 1\{F(j, n)F(k, n) > 0\}. \quad (26)$$

Then for each $\alpha < d$,

$$\mathbb{P}\{\dim_h(\eta[0, 1]) \geq \alpha\} > 0.$$

In particular, if it is known that there is a d_* such that $\mathbb{P}\{\dim_h(\eta[0, 1]) = d_*\} = 1$, then $d_* \geq d$.

Proof. Suppose $\alpha < d$. Let

$$r = \frac{d - \alpha}{d + \alpha} (1 - \xi)$$

By (26), there is a c_1 such that for $j < k$, if $F(j, n) F(k, n) > 0$,

$$\left| \eta\left(\frac{j}{n}\right) - \eta\left(\frac{k}{n}\right) \right|^{-d} \leq c_1 \left(\frac{n}{k - j} \right)^{1 - \xi + r},$$

and hence,

$$\left| \eta\left(\frac{j}{n}\right) - \eta\left(\frac{k}{n}\right) \right|^{-\alpha} \leq c_1 \left(\frac{n}{k - j} \right)^{1 - \xi - r}.$$

By (25), there is a c_2 such that

$$\mathbb{E}[F(j, n) F(k, n)] \leq c_2 \left(\frac{n}{k - j + 1} \right)^{\xi + (r/2)}.$$

The hypotheses of Lemma 3.3 hold with $\delta = \xi + (r/2)$, $\delta' = \xi + r$. □

Remark If the assumption (25) is strengthened to

$$\mathbb{E}[F(j, n) F(k, n)] \leq c \left(\frac{n}{|j - k| + 1} \right)^{\xi},$$

then the conclusion can be strengthened to

$$\mathbb{P}\{\dim_h(\eta[0, 1]) \geq d\} > 0.$$

Remark One important case of this lemma is when η is the identity function and $d = 1 - \xi$ in which case (26) is immediate.

It will be useful for us to give a slight generalization of Corollary 3.4. Corollary 3.4 is the particular case of Corollary 3.5 with $\eta(j, n) = \eta(j/n)$.

Corollary 3.5. *Suppose $\eta : [0, 1] \rightarrow \mathbb{R}^m$ is a random curve,*

$$\{F(j, n) : n = 1, 2, \dots, j = 1, 2, \dots, n\}$$

are nonnegative random variables, and

$$\{\eta(j, n) : n = 1, 2, \dots, j = 1, 2, \dots, n\}$$

are \mathbb{R}^m -valued random variables all defined on the same probability space. Suppose there exist $c < \infty$; and a subpower function ψ such that the following holds for $n = 1, 2, \dots$, and $1 \leq j \leq k \leq n$.

$$\frac{1}{c} \leq \frac{1}{n} \sum_{j=1}^n \mathbb{E}[F(j, n)] \leq c. \tag{27}$$

$$\mathbb{E}[F(j, n)F(k, n)] \leq \left(\frac{n}{k-j+1}\right)^\xi \psi\left(\frac{n}{k-j+1}\right), \quad (28)$$

$$|\eta(j, n) - \eta(k, n)|^d \geq \frac{1}{c} \left(\frac{|k-j|}{n}\right)^{1-\xi} \psi\left(\frac{n}{|j-k|+1}\right)^{-1} 1\{F(j, n)F(k, n) > 0\}, \quad (29)$$

and such that with probability one

$$\lim_{n \rightarrow \infty} \max \{\text{dist}[\eta(j, n), \eta[0, 1]] : j = 1, \dots, n\} = 0. \quad (30)$$

Then for each $\alpha < d$,

$$\mathbb{P}\{\dim_h(\eta[0, 1]) \geq \alpha\} > 0.$$

In particular, if it is known that there is a d_* such that $\mathbb{P}\{\dim_h(\eta[0, 1]) = d_*\} = 1$, then $d_* \geq d$.

Proof. The proof proceeds as in Lemma 3.3 and Corollary 3.4. The measure $\mu_{j,n}$ in the proof of Lemma 3.3 is placed on the disk centered at $\eta(j, n)$ rather than $\eta(j/n)$. The key observation is that on the event (30), any subsequential limit of the measures ν_n must be supported on $\eta[0, 1]$. \square

3.3 A lemma about conformal maps

In this section, we state a result about conformal maps that we will need. For $r \geq 1$, let

$$\mathcal{R}(r) = [-r, r] \times [1/r, r] = \{x + iy : -r \leq x \leq r, \quad 1/r \leq y \leq r\}.$$

The proof of the next lemma is simple and left to the reader. It is an elementary way of stating the fact that the diameter of $\mathcal{R}(r)$ in the hyperbolic metric is $O(\log r)$. We then use that lemma to prove another lemma which is essentially a corollary of the Koebe (1/4)-theorem and the distortion theorem (see [6, Section 3.2]) which we recall now. Suppose $f : D \rightarrow \mathbb{C}$ is a conformal transformation, $z \in D$, and $d = \text{dist}(z, \partial D)$. Then, $f(D)$ contains the open ball of radius $d|f'(z)|/4$ about $f(z)$ and

$$\frac{1-r}{(1+r)^3} |f'(z)| \leq |f'(w)| \leq \frac{1+r}{(1-r)^3} |f'(z)| \quad |w - z| \leq rd. \quad (31)$$

Lemma 3.6. *There is a $c_2 < \infty$ such that for every $r \geq 1$ and $z, w \in \mathcal{R}(r)$, we can find $z = z_0, z_1, \dots, z_k = w$ with $k \leq c_2 \log(r+1)$; $z_0, \dots, z_k \in \mathcal{R}(r)$; and such that for $j = 1, \dots, k$,*

$$|z_j - z_{j-1}| < \frac{1}{8} \max\{\text{Im}(z_j), \text{Im}(z_{j-1})\}.$$

Lemma 3.7. *There exists $\alpha < \infty$ such that if $g : \mathbb{H} \rightarrow \mathbb{C}$ is a conformal transformation, $r \geq 1$ and $z, w \in \mathcal{R}(r)$, then*

$$|g'(w)| \leq (2r)^\alpha |g'(z)|, \\ \text{dist}[g(w), \partial g(\mathbb{H})] \leq (2r)^\alpha \text{dist}[g(z), \partial g(\mathbb{H})].$$

Proof. The Koebe-(1/4) Theorem and the Distortion Theorem imply that there is a c such that if $z \in \mathbb{H}$ and $|w - z| \leq \text{Im}(z)/8$, then

$$c^{-1} |g'(z)| \leq |g'(w)| \leq c |g'(z)|$$

and

$$c^{-1} \text{dist}[g(z), \partial g(\mathbb{H})] \leq \text{dist}[g(w), \partial g(\mathbb{H})] \leq c \text{dist}[g(z), \partial g(\mathbb{H})].$$

Hence if $z, w \in \mathcal{R}(r)$ and z_0, \dots, z_k are as above,

$$|g'(w)| \leq c^k |g'(z)|$$

and

$$\text{dist}[g(w), \partial g(\mathbb{H})] \leq c^k \text{dist}[g(z), \partial g(\mathbb{H})].$$

But, $c^k \leq c^{c_2 \log(r+1)} = (r+1)^\alpha \leq (2r)^\alpha$, where $\alpha = c_2 \log c$. □

4 Proof

In this section we give the proof of the following theorem assuming one key estimate, Theorem 4.2, that we prove in Section 5.

Theorem 4.1. *If γ is an SLE_κ curve, $\kappa < 8$, then with probability one for all $t_1 < t_2$,*

$$\dim_h(\gamma[t_1, t_2]) = d.$$

We have already noted that there is a d_* such that with probability one for all $t_1 < t_2$, $\dim_h(\gamma[t_1, t_2]) = d_*$. The estimate $d_* \leq d$ is straightforward (see Section 6). The hard work is showing that $d_* \geq d$ and this is what we focus on. It suffices for us to consider $t_1 = 1, t_2 = 2$.

We will use the notation from Section 3.1. In order to show that $d_* \geq d$, we will show that the conditions of Corollary 3.5 are satisfied with

$$\xi = d(d-2) + 1 \in (0, 1).$$

As before, we set

$$\beta = d - \frac{3}{2} = \frac{\xi - 1}{d} + \frac{1}{2}.$$

For fixed positive integer n and integers $1 \leq j < k \leq n$, we let

$$S = S_{j,n} = 1 + \frac{j-1}{n}, \quad T = T_{j,k,n} = \frac{k-j}{n}, \quad S + T = 1 + \frac{k-1}{n},$$

$$\eta(t) = \gamma(1+t), \quad \eta(j, n) = \hat{f}_S(i/\sqrt{n}) = f_S(V_S + n^{-1/2}i).$$

Note that $1 \leq S \leq S + T \leq 2, 0 \leq T \leq 1$. We let $\mathcal{F} = \mathcal{F}_S$ denote the σ -algebra generated by $\{V_s : s \leq S\} = \{U_{T+s} - U_T : s \leq S\}$ and $\mathcal{G} = \mathcal{G}_{S,T}$ the σ -algebra generated by

$\{V_{S+t} - V_S : 0 \leq t \leq T\} = \{U_t : 0 \leq t \leq T\}$. Note that \mathcal{F} and \mathcal{G} are independent. We let $\mathcal{F} \vee \mathcal{G}$ be the σ -algebra generated by $\{U_t : 0 \leq t \leq S + T\}$.

Let us give an idea of the strategy. We will define $F(j, n) = n^{1-\frac{d}{2}} |\hat{f}'_S(i/\sqrt{n})|^d 1_E$ where $E = E_{j,n}$ is some event on which $|\hat{f}'_S(i/\sqrt{n})| \approx n^\beta$. The event E will describe “typical” behavior when we weight the paths by $|\hat{f}'_S(i/\sqrt{n})|^d$; in particular, it will satisfy

$$\mathbb{E} \left[|\hat{f}'_S(i/\sqrt{n})|^d 1_E \right] \asymp \mathbb{E} \left[|\hat{f}'_S(i/\sqrt{n})|^d \right].$$

To establish tightness and give lower bounds on Hausdorff dimension, we need to consider correlations which means estimating $\mathbb{E}[F(j, n)F(k, n)]$. Suppose, for example, $j = k$. If we did not include the event E then we would be estimating

$$\mathbb{E} \left[|\hat{f}'_S(i/\sqrt{n})|^{2d} \right],$$

which is not of the same order of magnitude as $(\mathbb{E}|\hat{f}'_S(i/\sqrt{n})|^d)^2$. Indeed, if we weight paths by $|\hat{f}'_S(i/\sqrt{n})|^{2d}$, we do not concentrate on paths with $|\hat{f}'_S(i/\sqrt{n})| \approx n^\beta$ but rather on paths with $|\hat{f}'_S(i/\sqrt{n})| \approx n^{\beta'}$ for some $\beta' > \beta$. However, when we include the 1_E term, we can write roughly

$$\mathbb{E}[(|\hat{f}'_S(i/\sqrt{n})|^d 1_E)^2] \approx n^{\beta d} \mathbb{E}[|\hat{f}'_S(i/\sqrt{n})|^d 1_E].$$

Remark The notation might be a little confusing. The time S will always be of order 1. The time T is at most that but generally will be much smaller.

4.1 Defining the $F(j, n)$

We define $F(j, n)$ to be the \mathcal{F} -measurable random variable

$$F(j, n) = n^{1-\frac{d}{2}} \left| \hat{f}'_S(i/\sqrt{n}) \right|^d 1_{E_{j,n}} \quad (32)$$

for some \mathcal{F} -measurable event

$$E_{j,n} = E_{j,n,1} \cap \cdots \cap E_{j,n,6},$$

which we will define now. We define $F(k, n)$ similarly; it will be $(\mathcal{F} \vee \mathcal{G})$ -measurable.

The event $E_{j,n}$ is defined in terms of the solution of the time-reversed Loewner equation. Let h_t, \tilde{h}_t as in Lemma 3.2. We write $Z_t = h_t(i/\sqrt{n}) - U_t = X_t + iY_t$, $\tilde{Z}_t = \tilde{h}_t(i/\sqrt{n}) - \tilde{U}_t = \tilde{X}_t + i\tilde{Y}_t$. In particular,

$$\begin{aligned} h_{s+T}(i/\sqrt{n}) &= \tilde{h}_s(Z_T) + U_T, \\ h'_{s+T}(i/\sqrt{n}) &= \tilde{h}'_s(Z_T) h'_T(i/\sqrt{n}). \end{aligned}$$

Remark The transformation h_T is \mathcal{G} -measurable and the transformation \tilde{h}_s is \mathcal{F} -measurable. The random variable Z_T is \mathcal{G} -measurable. The random variable $\tilde{h}'_s(Z_T)$ is neither \mathcal{F} -measurable nor \mathcal{G} -measurable. The key to bounding correlations at times S and $S + T$ is handling this random variable.

We will study h_t in detail in Section 5. In this section, we will need a couple of very simple estimates that follows immediately from the Loewner equation. For fixed z , $Y_t(z)$ is increasing in t and

$$\partial_t Y_t(z) = \frac{a Y_t(z)}{|Z_t(z)|^2}, \quad \partial_t Y_t(z)^2 = \frac{2a Y_t(z)^2}{X_t(z)^2 + Y_t(z)^2}. \quad (33)$$

In particular,

$$\operatorname{Im}(z)^2 + 2at \min_{0 \leq s \leq t} \frac{Y_t^2(z)}{|Z_t(z)|^2} \leq Y_t(z)^2 \leq \operatorname{Im}(z)^2 + 2at. \quad (34)$$

Also,

$$|h_t(z) - z| \leq \frac{at}{\operatorname{Im}(z)}, \quad |\log |h'_t(z)|| \leq \frac{at}{\operatorname{Im}(z)^2}. \quad (35)$$

The six events will depend on a subpower function ϕ_0 to be determined later. Given ϕ_0 we define the following events.

$$E_{k,n,1} = \left\{ Y_t \geq t^{\frac{1}{2}} \phi_0(1/t)^{-1} \text{ for all } 1/n \leq t \leq S+T \right\}. \quad (36)$$

$$E_{k,n,2} = \left\{ Y_t \geq t^{\frac{1}{2}} \phi_0(nt)^{-1} \text{ for all } 1/n \leq t \leq S+T \right\}. \quad (37)$$

$$E_{k,n,3} = \left\{ |X_t| \leq t^{\frac{1}{2}} \phi_0(1/t) \text{ for all } 1/n \leq t \leq S+T \right\},$$

$$E_{k,n,4} = \left\{ |X_t| \leq t^{\frac{1}{2}} \phi_0(nt) \text{ for all } 1/n \leq t \leq S+T \right\}.$$

$$E_{k,n,5} = \left\{ (nt)^\beta \phi_0(nt)^{-1} \leq |h'_t(i/\sqrt{n})| \leq (nt)^\beta \phi_0(nt) \text{ for all } 1/n \leq t \leq S+T \right\}$$

$$E_{k,n,6} = \left\{ t^{-\beta} \phi_0(1/t)^{-1} \leq \left| \frac{h'_{S+T}(i/\sqrt{n})}{h'_t(i/\sqrt{n})} \right| \leq t^{-\beta} \phi_0(1/t) \text{ for all } 1/n \leq t \leq S+T \right\}. \quad (38)$$

$E_{j,n,\cdot}$ are defined in the same way replacing $h_t, Z_t, S+T$ with $\tilde{h}_t, \tilde{Z}_t, S$.

Remark What we would really like to do is define is an event of the form

$$Y_t \asymp t^{\frac{1}{2}}, \quad |X_t| \leq c t^{\frac{1}{2}}, \quad |h'_t(i/\sqrt{n})| \asymp (nt)^\beta,$$

for all $0 \leq t \leq S+T$. However, this is too strong a restriction if we want the event to have positive probability (in the weighted measure). What we have done is modify this so that quantities are comparable for times near zero and for times near $S+T$ but the error may be larger for times in between (but still bounded by a subpower function).

Theorem 4.2. *There exist c_1, c_2 such that for all $t \geq 1/n$,*

$$\mathbb{E} \left[\left| h'_t(i/\sqrt{n}) \right|^d \right] = \mathbb{E} \left[\left| h'_{tn}(i) \right|^d \right] \leq c_2 (tn)^{\frac{d}{2}-1}. \quad (39)$$

Moreover there exists a power function ϕ_0 such that if $E_{j,n}$ is defined as above, then

$$\mathbb{E} \left[\left| \tilde{h}'_S(i/\sqrt{n}) \right|^d 1_{E_{j,n}} \right] \geq c_1 n^{\frac{d}{2}-1}. \quad (40)$$

Proof. See Section 5. □

Remark The equality in (39) follows immediately from scaling.

4.2 Handling the correlations

Theorem 4.2 discusses the function \tilde{h}_t and the corresponding processes \tilde{X}_t, \tilde{Y}_t for a fixed value of S . In this section we assume Theorem 4.2 and show how to verify the hypotheses of Corollary 3.4 for ξ as defined earlier and some subpower function ϕ . Here $F(j, n)$ is defined as in (32). The first hypothesis (27) follows immediately from (39) so we will only need to consider (28)–(30). Throughout this subsection ϕ will denote a subpower function, but its value may change from line to line.

4.2.1 The estimate (28)

We first consider $j = k$. Then

$$\mathbb{E}[F(j, n)^2] = n^{2-d} \mathbb{E} \left[\left| \tilde{h}'_S(i/\sqrt{n}) \right|^{2d} 1_{E_{j,n}} \right].$$

But on the event $E_{j,n}$ we know that $|\tilde{h}'_S(i/\sqrt{n})| \leq n^\beta \phi(n)$. Therefore, using (39),

$$\mathbb{E}[F(j, n)^2] \leq n^{2-d+\beta d} \mathbb{E} \left[\left| \tilde{h}'_S(i/\sqrt{n}) \right|^d \right] \phi(n) \leq n^\xi \phi(n).$$

We now assume $j < k$. We need to give an upper bound for

$$\mathbb{E}[F(j, n) F(k, n)] = n^{2-d} \mathbb{E} \left[1_{E_{j,n}} |\tilde{h}'_S(i/\sqrt{n})| 1_{E_{k,n}} |h'_{S+T}(i/\sqrt{n})| \right].$$

Let $\tilde{E}_{k,n} = \tilde{E}_{k,n,1} \cap \tilde{E}_{k,n,3}$ where $\tilde{E}_{k,n,j}$ is defined as $E_{k,n,j}$ except that $1/n \leq t \leq S+T$ is replaced with $1/n \leq t \leq T$. Then $\tilde{E}_{k,n}$ is \mathcal{G} -measurable and $E_{k,n} \subset \tilde{E}_{k,n}$. From (36), we can write

$$h'_{T+S}(i/\sqrt{n}) = h'_T(i/\sqrt{n}) \tilde{h}'_S(Z_T).$$

Therefore,

$$n^{d-2} \mathbb{E}[F(j, n) F(k, n)] \leq$$

$$\mathbb{E} \left[1_{E_{j,n}} |\tilde{h}'_S(i/\sqrt{n})| |\tilde{h}'_S(Z_T)| 1_{\tilde{E}_{k,n}} |h'_T(i/\sqrt{n})| \right]. \quad (41)$$

This is the expectation of a product of five random variables. The first two are \mathcal{F} -measurable and the last two are \mathcal{G} -measurable. The middle random variable $|\tilde{h}'_S(Z_T)|$ uses information from both σ -algebras: the transformation \tilde{h}_S is \mathcal{F} -measurable but it is evaluated at Z_T which is \mathcal{G} -measurable.

We claim that it suffices to show that on the event $E_{j,n} \cap \tilde{E}_{k,n}$,

$$\left| \tilde{h}'_S(Z_T) \right|^d \leq T^{-\beta d} \phi(1/T), \quad (42)$$

for some subpower function ϕ . Indeed, once we have established this we can see that the expectation in (41) is bounded above by

$$T^{-\beta d} \phi(1/T) \mathbb{E} \left[|\tilde{h}'_S(i/\sqrt{n})| |h'_T(i/\sqrt{n})| \right],$$

which by independence equals

$$T^{-\beta d} \phi(1/T) \mathbb{E} \left[|\tilde{h}'_S(i/\sqrt{n})| \right] \mathbb{E} \left[|h'_T(i/\sqrt{n})| \right].$$

Using (39), we then have that this is bounded by

$$T^{-\beta d} \phi(T) n^{\frac{d}{2}-1} (nT)^{\frac{d}{2}-1} = T^{-\xi} \phi(1/T) = \left(\frac{n}{k-j} \right)^\xi \phi \left(\frac{n}{k-j} \right).$$

Hence, we only need to establish (42).

On the event $\tilde{E}_{k,n}$, we know that

$$|X_T| \leq T^{1/2} \phi(1/T), \quad \phi(1/T)^{-1} T^{1/2} \leq Y_T \leq c T^{1/2}. \quad (43)$$

Using Lemma 3.7 and (43), we can see that

$$|\tilde{h}'_S(Z_T)|^d \leq \phi(1/T) |\tilde{h}'_S(i\sqrt{T})|^d.$$

We can write

$$\tilde{h}'_S(i\sqrt{T}) = \hat{h}'_{S-T}(\tilde{Z}_T(i\sqrt{T})) \tilde{h}'_T(i\sqrt{T}),$$

where \hat{h} is defined like h, \tilde{h} except using $S - T$ in place of $S + T, S$. By (35), we know that

$$|\tilde{h}_T(i\sqrt{T})| \leq c \sqrt{T}, \quad |\tilde{h}'_T(i\sqrt{T})| \leq c.$$

But on the event $E_{j,n}$,

$$|\tilde{X}_T| \leq T^{1/2} \phi(1/T), \quad T^{1/2} \phi(1/T)^{-1} \leq \tilde{Y}_T \leq c T^{1/2},$$

from which we can see that $|\tilde{Z}_T(i\sqrt{T})| \leq T^{1/2} \phi(1/T)$. Using Lemma 3.7 again we see that

$$|\hat{h}'_{S-T}(\tilde{Z}_T(i\sqrt{T}))| \leq \phi(1/T) |\hat{h}'_{S-T}(Z_{2T})|,$$

and the right-hand side is bounded by $\phi(1/T) T^{-\beta}$ by (38). This gives (42).

4.2.2 The estimate (29)

Assume that $j < k$ and that $E_{j,n}$ and $E_{k,n}$ both occur. Then,

$$\eta(k, n) = \hat{f}_{S+T}(i/\sqrt{n}) = h_{S+T}(1/\sqrt{n}) + U_{T+S} = \tilde{h}_S(Z_T) + U_S.$$

Therefore,

$$\eta(k, n) - \eta(j, n) = \tilde{h}(Z_S) - \tilde{h}_S(i/\sqrt{n}).$$

By (36) and (37) we know that

$$Y_T \geq T^{\frac{1}{2}} \max\{\phi(nT)^{-1}, \phi(1/T)^{-1}\} = T^{\frac{1}{2}} \max\left\{\phi(k-j)^{-1}, \phi\left(\frac{n}{k-j}\right)^{-1}\right\}.$$

Also, using (34), we see that there is a c such that

$$Y_T - n^{-1/2} > c Y_T. \quad (44)$$

Also,

$$h'_{S+T}(i/\sqrt{n}) = h'_T(i/\sqrt{n}) \tilde{h}'_S(Z).$$

By (38) we know that

$$|\tilde{h}'_S(Z_T)| \geq T^{-\beta} \phi\left(\frac{n}{k-j}\right)^{-1}.$$

The Koebe (1/4)-Theorem tells us that the image of the ball of radius cY_T about Z_T under \tilde{h}_S contains a ball \mathcal{B} of radius

$$\frac{cY_T}{4} |\tilde{h}'(Z_T)| \geq cT^{\frac{1}{2}+\beta} \phi\left(\frac{n}{k-j}\right)^{-1}$$

about $\eta(k, n)$. But (44) tells us that i/\sqrt{n} is not in the ball of radius cYT about ZT and hence $\eta(j, n) \notin \mathcal{B}$. Therefore,

$$|\eta(k, n) - \eta(j, n)|^d \geq T^{\frac{d}{2}+d\beta} \phi\left(\frac{n}{k-j}\right)^{-1} = T^{\xi-1} \phi\left(\frac{n}{k-j}\right)^{-1}.$$

Remark Note that we do not expect that last estimate to hold for all k, j , especially for $\kappa > 4$ for which SLE_κ has double points. The restriction to the event $E_{j,n} \cap E_{k,n}$ is a major restriction.

4.2.3 The estimate (30)

This estimate follows from (7). This lemma was essentially proved by Rohde and Schramm when they proved existence of the curve. We rederive this result in Section 7.

Remark In fact, we do not need this to prove our result. On the event $E_{j,n}$, we have $|\hat{f}'_S(i/\sqrt{n})| \approx n^\beta$. Therefore, using the Koebe (1/4)-Theorem we can conclude on this event that for every $\epsilon > 0$, there is a c such that

$$\text{dist} \left[\hat{f}_S(i/\sqrt{n}), \gamma[0, s] \cap \mathbb{R} \right] \leq c n^{\beta+\epsilon} n^{-1/2} = c n^{d+\epsilon-2}.$$

Since $d < 2$, this goes to 0 for ϵ sufficiently small.

5 Derivative estimates for reverse flow

Theorem 4.2 is a statement about the flow $h_t(z)$ for a fixed $z \in \mathbb{H}$. By appropriate change of variables, this flow can be understood in terms of a one-dimensional diffusion. Throughout this section we fix a . If $a > 1/4$, we let d, β, ξ be as above. All constants in this section may depend on a . We will consider solutions of the time-reversed Loewner equation

$$\dot{h}_t(z) = \frac{a}{U_t - h_t(z)}, \quad h_0(z) = z, \quad (45)$$

where $U_t = -B_t$ is a standard Brownian motion. The scaling properties of Brownian motion imply that for each $r > 0$, the distribution of the random function $z \mapsto r^{-1} h_{r^2 t}(rz)$ is the same as the distribution of $z \mapsto h_t(z)$. In particular, the distribution of $h'_{r^2 t}(rz)$ is the same as that of $h'_t(z)$. We define $X_t(z), Y_t(z)$ by

$$h_t(z) - U_t = X_t(z) + iY_t(z).$$

We now restate Theorem 4.2 in a scaled form.

Theorem 5.1. *Suppose $a > 1/4$. There exist $0 < c_1, c_2 < \infty$ and a subpower function ϕ such that the following holds. Let $X_t = X_t(i), Y_t = Y_t(i)$, and let $E = E(\phi, t)$ be the event that for all $1 \leq s \leq t$,*

$$\begin{aligned} \sqrt{s} \max \left\{ \frac{1}{\phi(s)}, \frac{1}{\phi(t/s)} \right\} &\leq Y_s \leq \sqrt{2as + 1}, \\ \frac{s^\beta}{\phi(s)} &\leq |h'_s(i)| \leq s^\beta \phi(s), \quad \frac{(t/s)^\beta}{\phi(t/s)} \leq \frac{|h'_t(i)|}{|h'_s(i)|} \leq (t/s)^\beta \phi(t/s), \\ |X_s| &\leq \sqrt{s} \min \{ \phi(s), \phi(t/s) \}. \end{aligned}$$

Then, for all $t \geq 1$,

$$c_1 t^{\frac{d}{2}-1} \leq \mathbb{E} \left[|h'_t(i)|^d 1_E \right] \leq \mathbb{E} \left[|h'_t(i)|^d \right] \leq c_2 t^{\frac{d}{2}-1}. \quad (46)$$

We will prove a more general result than this theorem. Theorem 5.1 is a special case of Propositions 5.14 and 5.19 which treat the upper and lower bounds, respectively. Careful examination of the proofs will show that one can choose

$$\phi(x) = C \exp \{ [\log(x+1)]^{1/2} [\log \log(x+2)]^u \}$$

for some $C, u < \infty$. However, we do not need the exact form of ϕ in this paper and we will not keep track of this.

Before going into details, let us give a quick overview of the strategy. The basic idea is to consider the following martingale

$$M_t = |h'_t(i)|^d Y_t^{2-d} \sqrt{R_t^2 + 1}, \quad R_t = X_t/Y_t.$$

The first two terms in M_t are (random) differentiable functions of t ; only the third has nontrivial quadratic variation. We expect that typically $Y_t \approx t^{1/2}$ and $R_t^2 + 1 \approx 1$. This suggests but does not prove the estimate $\mathbb{E}[|h'_t(i)|^d] \asymp t^{\frac{d}{2}-1}$. To estimate something like $\mathbb{E}[|h'_t(i)|^d 1_E]$, we consider a similar expectation $\mathbb{E}[M_t 1_E]$. The Girsanov theorem tells us that we can compute expectations like these by computing the probability of E under a different measure which corresponds to changing the SDE by adding to the drift. In our case, the derived equation is fairly simple to analyze and this allows us to prove the theorem. The computation is made somewhat simpler by changing time so that Y_t grows deterministically. This time change was used by Rohde and Schramm ([11]) and analogues of it have appeared in many places. This time parametrization works very well when considering $h_t(z)$ for a single z , which is what we are doing in this section. However, the time change depends on z , so it is not so easy to use it for studying the joint distribution of $(h_t(z), h_t(w))$ for distinct z, w .

Sections 5.2 and 5.6.1 prove results that will be used in [10]. These sections can be skipped if the reader is only interested in Theorem 5.1. Also, Theorem 5.1 needs Proposition 5.14 only for $x = 0$ for which the proof simplifies.

5.1 An important martingale

Here we do some straightforward computations and introduce the important martingales for the reverse Loewner flow. Let $z = y(x+i) \in \mathbb{H}$. Let $Z_t = Z_t(z) = h_t(z) - U_t = X_t + iY_t$. The time-reversed Loewner equation (45) can be written as

$$dX_t = -\frac{aX_t}{X_t^2 + Y_t^2} dt + dB_t, \quad \partial_t Y_t = \frac{aY_t}{X_t^2 + Y_t^2} = Y_t \frac{a(X_t^2 + Y_t^2)}{(X_t^2 + Y_t^2)^2}. \quad (47)$$

Note that Y_t is strictly increasing and $\partial_t(Y_t^2) \leq 2a$ which implies

$$Y_t^2 \leq 2at + y^2. \quad (48)$$

Differentiating (45) with respect to z gives

$$\partial_t [\log h'_t(z)] = \frac{a}{Z_t^2}, \quad h'_t(z) = \exp \left\{ \int_0^t \frac{a}{Z_s^2} ds \right\}.$$

$$|h'_t(z)| = \exp \left\{ \int_0^t \operatorname{Re}[aZ_s^{-2}] ds \right\} = \exp \left\{ \int_0^t \frac{a(X_s^2 - Y_s^2)}{(X_s^2 + Y_s^2)^2} ds \right\},$$

$$\partial_t |h'_t(z)| = |h'_t(z)| \frac{a(X_t^2 - Y_t^2)}{(X_t^2 + Y_t^2)^2}. \quad (49)$$

Let

$$\Psi_t = \frac{|h'_t(z)|}{Y_t}, \quad R_t = \frac{X_t}{Y_t}.$$

Itô's formula and the chain rule give the following.

$$d(R_t^2 + 1)^{r/2} = (R_t^2 + 1)^{r/2} \left[\frac{(-2ar + \frac{r^2}{2} - \frac{r}{2})X_t^2 + \frac{r}{2}Y_t^2}{(X_t^2 + Y_t^2)^2} dt + \frac{rX_t}{X_t^2 + Y_t^2} dB_t \right], \quad (50)$$

$$\partial_t [\Psi_t^k] = \Psi_t^k \frac{-2ak Y_t^2}{(X_t^2 + Y_t^2)^2}.$$

In particular Ψ_t decreases in t which combined with (48) gives

$$|h'_t(z)| \leq y^{-1} Y_t(z) \leq \sqrt{2a(t/y^2)} + 1. \quad (51)$$

Proposition 5.2. *If $r, \theta, \lambda \in \mathbb{R}, z \in \mathbb{H}$ and*

$$N_t = |h'_t(z)|^\lambda Y_t^{\frac{\theta}{a} - \lambda} (R_t^2 + 1)^{\frac{r}{2}} = \Psi_t^\lambda Y_t^{\frac{\theta}{a}} (R_t^2 + 1)^{\frac{r}{2}}.$$

Then,

$$dN_t = N_t \left[\frac{j_X X_t^2 + j_Y Y_t^2}{(X_t^2 + Y_t^2)^2} dt + \frac{r X_t}{X_t^2 + Y_t^2} dB_t \right], \quad (52)$$

where

$$j_X = \frac{r^2}{2} - \left(2a + \frac{1}{2} \right) r + \theta,$$

$$j_Y = \frac{r}{2} - 2a\lambda + \theta.$$

If $r \in \mathbb{R}$, and

$$\lambda = \lambda(r) = r \left(1 + \frac{1}{2a} \right) - \frac{r^2}{4a},$$

then

$$M_t = \Psi_t^\lambda Y_t^{2\lambda - \frac{r}{2a}} (R_t^2 + 1)^{r/2} = |h'_t(z)|^\lambda Y_t^{r - \frac{r^2}{4a}} (R_t^2 + 1)^{r/2}, \quad (53)$$

is a martingale satisfying

$$dM_t = \frac{r X_t}{X_t^2 + Y_t^2} M_t dB_t. \quad (54)$$

Proof. The computations (52) – (54) are straightforward using the product rule and Itô's formula. Therefore, M_t is a nonnegative local martingale satisfying (54). We will now show how we can use the Girsanov theorem to conclude that M_t is a martingale. Suppose M_t is a nonnegative, continuous local martingale satisfying

$$dM_t = J_t M_t dB_t,$$

and, for ease, assume that $M_0 = 1$. For any $n \geq 1$, let $\tau_n = \inf\{t : M_t \geq n\}$. Then $M_t^{(n)} := M_{t \wedge \tau_n}$ is a uniformly bounded martingale satisfying

$$dM_t^{(n)} = J_t^{(n)} M_t^{(n)} dB_t,$$

where $J_t^{(n)} = J_t 1_{\{\tau_n > t\}}$. Let $Q_t^{(n)}$ be the probability measure given by weighting by $M_t^{(n)}$, i.e., for every \mathcal{F}_t -measurable event A ,

$$Q_t^{(n)}[A] = \mathbb{E} \left[M_t^{(n)} 1_A \right].$$

The Girsanov theorem tells us that

$$W_t^{(n)} = B_t - \int_0^t J_s^{(n)} ds,$$

is a standard Brownian motion with respect to the measure $Q_t^{(n)}$, or equivalently,

$$dB_t = J_t^{(n)} dt + dW_t^{(n)}. \quad (55)$$

The same holds for $M_t = M_t^{(\infty)}$ provided that M_t is a martingale. In order to see whether or not M_t is a martingale, one needs to check whether or not some mass “disappears” as $n \rightarrow \infty$. If for every fixed t ,

$$\lim_{n \rightarrow \infty} Q_t^{(n)}\{\tau_n \leq t\} = \lim_{n \rightarrow \infty} \mathbb{E}[M_{\tau_n} ; \tau_n \leq t] = 0,$$

then no mass disappears. Since the stopped process satisfies the equation (55), we can see that M_t is a martingale provided that the system

$$dB_t = J_t dt + dW_t, \quad dM_t = J_t M_t dB_t, \quad (56)$$

has no explosion in finite time, i.e., for each $\epsilon > 0, t < \infty$, there is an $N < \infty$ such that if B_t, M_t satisfies (56), then with probability at least $1 - \epsilon$, $\sup_{s \leq t} (|B_s| + M_s) \leq N$. In order to simplify the notation, when we weight by the local martingale, then we will say that Girsanov theorem implies that B_t satisfies

$$dB_t = J_t dt + dW_t.$$

To be precise, this should be interpreted in terms of stopping times as in (55).

We now consider the case at hand. If we use Girsanov's theorem and weight B_t by the martingale M_t as in (53), then

$$dB_t = \frac{r X_t}{X_t^2 + Y_t^2} dt + d\tilde{B}_t$$

where \tilde{B}_t is a Brownian motion with respect to the new measure. In other words, with respect to the new measure

$$dX_t = \frac{(r-a) X_t}{X_t^2 + Y_t^2} dt + d\tilde{B}_t. \quad (57)$$

Note that Y_t and $|h'_t(z)|^d$ are differentiable quantities so their equations do not change in the new measure. By comparing to a Bessel equation, it is easy to check that if X_t satisfies the above equation then there is no explosion in finite time. Since

$$M_t = \Psi_t^\lambda Y_t^{2\lambda - \frac{r}{2a}} (R_t^2 + 1)^{r/2} = \Psi_t^\lambda Y_t^{2\lambda - \frac{r}{2a} - r} (X_t^2 + Y_t^2)^{r/2},$$

and $\Psi_t \leq 1/y$, $Y_t \leq \sqrt{2at + y^2}$, this also shows that M_t has no explosion in finite time. This is enough to conclude that M_t is a martingale. \square

The particular values of the parameter that are most important for Theorem 5.1 are

$$r = 1, \quad \lambda = d, \quad r - \frac{r^2}{4a} = 2 - d.$$

5.2 Simple consequences

Proposition 5.2 suggests but does not prove the upper bound in Theorem 5.1. We have

$$\mathbb{E} \left[|h'_t(i)|^\lambda Y_t^{r - \frac{r^2}{4a}} (R_t^2 + 1)^{r/2} \right] = 1.$$

One might hope that the typical value of $R_t^2 + 1$ is of order 1 and the typical value of Y_t is of order \sqrt{t} . If this were true then the expectation would be comparable to

$$\mathbb{E} \left[|h'_t(i)|^\lambda (\sqrt{t})^{r - \frac{r^2}{4a}} \right],$$

and we would have the upper bound. In Proposition 5.14 we show this is true for a range of positive r . The range includes $r = 1$ if $a > 1/4$.

In this section we consider the simpler problem of giving upper bounds for

$$\mathbb{E} \left[|h'_t(z)|^\lambda; Y_t \geq \epsilon \sqrt{t} \right].$$

The results in this section will be used in [10]. By scaling, we see that

$$\mathbb{E} \left[|h'_t(y(x+i))|^\lambda; Y_t(y(x+i)) \geq \epsilon \sqrt{t} \right] = \mathbb{E} \left[|h'_{t/y^2}(x+i)|^\lambda; Y_t(x+i) \geq \epsilon \sqrt{t/y^2} \right],$$

so it suffices to consider the case $z = x + i$. We restrict our consideration to $0 \leq r \leq 2a + 1$. In this interval, $r \mapsto \lambda(r)$ is one-to-one and we can write the martingale (53) as

$$M_t = |h'_t(z)|^\lambda Y_t^{\zeta(\lambda)} (R_t^2 + 1)^{r(\lambda)},$$

where

$$r = r(\lambda) = 2a + 1 - 2a \sqrt{\left(1 + \frac{1}{2a}\right)^2 - \frac{\lambda}{a}}, \quad (58)$$

$$\zeta = \zeta(\lambda) = r - \frac{r^2}{4a} = \lambda - \frac{r}{2a} = \lambda + \sqrt{\left(1 + \frac{1}{2a}\right)^2 - \frac{\lambda}{a}} - 1 - \frac{1}{2a}. \quad (59)$$

Note that $\lambda \mapsto \zeta(\lambda)$ is strictly concave. We define λ_c by $\zeta'(\lambda_c) = -1$; in other words,

$$r(\lambda_c) = 2a + \frac{1}{2}, \quad \lambda_c = a + \frac{3}{16a} + 1, \quad \zeta(\lambda_c) = a - \frac{1}{16a}.$$

Proposition 5.3. *Suppose $0 \leq \lambda \leq \lambda_c = a + \frac{3}{16a} + 1$. Then for every $x \in \mathbb{R}, \epsilon > 0$, if $Y_t = Y_t(x + i)$,*

$$\mathbb{E} \left[|h'_t(x + i)|^\lambda; Y_t \geq \epsilon \sqrt{t} \right] \leq (x^2 + 1)^{r/2} \epsilon^{-\zeta} t^{-\zeta/2},$$

where r, ζ are as defined in (58) and (59).

Proof.

$$\begin{aligned} \mathbb{E} \left[|h'_t(x + i)|^\lambda; Y_t \geq \epsilon \sqrt{t} \right] &\leq \epsilon^{-\zeta} t^{-\zeta/2} \mathbb{E} \left[|h'_t(x + i)|^\lambda Y_t^\zeta \right] \\ &\leq \epsilon^{-\zeta} t^{-\zeta/2} \mathbb{E}[M_t] \\ &= \epsilon^{-\zeta} t^{-\zeta/2} (x^2 + 1)^{r/2}. \end{aligned}$$

□

The preceding proof did not use the fact that $\lambda \leq \lambda_c$. However, for $\lambda > \lambda_c$ we can get a better estimate as given in the next proposition.

Proposition 5.4. *Suppose $\delta > 0$. Then, for every $x \in \mathbb{R}, \epsilon > 0$, if $Y_t = Y_t(x + i)$,*

$$\mathbb{E} \left[|h'_t(x + i)|^{a + \frac{3}{16a} + 1 + \delta}; Y_t \geq \epsilon \sqrt{t} \right] \leq (2at + 1)^{\frac{\delta}{2}} \epsilon^{\frac{1}{16a} - a} t^{\frac{1}{32a} - \frac{a}{2}} (x^2 + 1)^{a + \frac{1}{4}}.$$

Proof. We know from (51) that $|h'_t(x + i)| \leq \sqrt{2at + 1}$. Therefore,

$$\mathbb{E} \left[|h'_t(x + i)|^{\lambda_c + \delta}; Y_t \geq \epsilon \sqrt{t} \right] \leq (2at + 1)^{\delta/2} \mathbb{E} \left[|h'_t(x + i)|^{\lambda_c}; Y_t \geq \epsilon \sqrt{t} \right].$$

□

Remark Roughly speaking, we have shown

$$\mathbb{E} \left[|h'_t(i)|^\lambda; Y_t \geq \epsilon \sqrt{t} \right] \leq c(\epsilon) t^{\beta(\lambda)},$$

where

$$2\beta(\lambda) = \begin{cases} -\zeta(\lambda), & \lambda \leq \lambda_c, \\ -\zeta(\lambda_c) + (\lambda - \lambda_c), & \lambda \geq \lambda_c. \end{cases}$$

Since ζ is strictly concave with $\zeta'(\lambda_c) = -1$,

$$-\zeta(\lambda_c) + (\lambda - \lambda_c) < -\zeta(\lambda), \quad \lambda > \lambda_c.$$

This is a standard “multifractal” analysis of a moment. As λ increases to λ_c , the expectation of $|h'_t(i)|^\lambda$ concentrates on the event that $|h'_t(i)| \approx t^{-\zeta'(\lambda)/2}$. When λ reaches λ_c , the expectation concentrates on the event $|h'_t(i)| \approx t^{1/2}$. However, we know that $|h'_t(i)| \leq ct^{1/2}$ so all higher powers λ also concentrate on this event. This makes ζ a linear function for $\lambda \geq \lambda_c$.

Example In [10], we will need to consider the case $a > 1/4, \lambda = 2d = 2 + \frac{1}{2a}$. We will consider two subcases.

- $5/4 \leq a < \infty$. In this range $2d \leq \lambda_c$. We have

$$\begin{aligned} r = r(2d) &= 2a + 1 - 2a \sqrt{1 - \frac{1}{a} - \frac{1}{4a^2}}, \\ \zeta &= \zeta(2d) = 1 + \sqrt{1 - \frac{1}{a} - \frac{1}{4a^2}}, \\ \mathbb{E} \left[|h'_t(x+i)|^{2d}; Y_t \geq \epsilon \sqrt{t} \right] &\leq c_\epsilon (x^2 + 1)^{r/2} t^{-\zeta/2}. \end{aligned} \tag{60}$$

- $1/4 < a \leq 5/4$. In this range

$$2d = \lambda_c + \left[1 + \frac{5}{16a} - a \right],$$

where the term in brackets is nonnegative, and hence

$$\mathbb{E} \left[|h'_t(x+i)|^{2d}; Y_t \geq \epsilon \sqrt{t} \right] \leq c_\epsilon (x^2 + 1)^{a+\frac{1}{4}} (t+1)^{-\tilde{\zeta}/2} \tag{61}$$

where

$$\tilde{\zeta} = \zeta(\lambda_c) - \left[1 + \frac{5}{16a} - a \right] = 2a - \frac{3}{8a} - 1.$$

Note that $\tilde{\zeta} = 0$ if $a = 3/4$ and $\tilde{\zeta} < 0$ for $1/4 < a < 3/4$.

5.3 Change of time

It is convenient to change time so that $\log Y_t$ grows linearly. This converts the two-variable process (X_s, Y_s) into a one-variable process. Let

$$\sigma(t) = \inf\{s : Y_s = e^{at}\}, \quad \hat{Y}_t = Y_{\sigma(t)} = e^{at}, \quad \hat{X}_t = X_{\sigma(t)}, \quad K_t = R_{\sigma(t)} = e^{-at} \hat{X}_t.$$

Lemma 5.5.

$$\dot{\sigma}(t) = \hat{X}_t^2 + \hat{Y}_t^2 = \hat{X}_t^2 + e^{2at}, \quad \sigma(t) = \int_0^t (\hat{X}_s^2 + e^{2as}) ds = \int_0^t e^{2as} (K_s^2 + 1) ds. \quad (62)$$

Proof. Since $\partial_t \hat{Y}_t = a \hat{Y}_t$, we have by (47),

$$a \hat{Y}_t = \partial_t \hat{Y}_t = \dot{Y}_{\sigma(t)} \dot{\sigma}(t) = \frac{a \hat{Y}_t}{\hat{X}_t^2 + \hat{Y}_t^2} \dot{\sigma}(t).$$

□

Using (47) we get

$$\begin{aligned} d\hat{X}_t &= -a \hat{X}_t dt + \sqrt{\hat{X}_t^2 + e^{2at}} dB_t, \\ dK_t &= -2a K_t dt + \sqrt{K_t^2 + 1} dB_t. \end{aligned} \quad (63)$$

From (49) we see that

$$\partial_t |h'_{\sigma(t)}(z)| = |h'_{\sigma(t)}(z)| \frac{a(\hat{X}_t^2 - \hat{Y}_t^2)}{\hat{X}_t^2 + \hat{Y}_t^2} = a |h'_{\sigma(t)}(z)| \left[1 - \frac{2}{K_t^2 + 1} \right],$$

and hence,

$$e^{-at} |h'_{\sigma(t)}(z)| = \exp \left\{ -a \int_0^t \frac{2 ds}{K_s^2 + 1} \right\}.$$

5.4 The SDE (63)

We will study the equation (63) from last subsection which we write as

$$dK_t = \left(\frac{1}{2} - q - r \right) K_t dt + \sqrt{K_t^2 + 1} dB_t, \quad (64)$$

where $q = 2a + \frac{1}{2} - r$. In this section, we view q, r as the given parameters and we define a by $2a = q + r - \frac{1}{2}$. We will be primarily interested in $q > 0$. Note that

$$q > 0 \iff r < 2a + \frac{1}{2}.$$

We write $\mathbb{E}^x, \mathbb{P}^x$ for expectations and probabilities assuming $K_0 = x$. If the x is omitted, then it is assumed that $K_0 = 0$.

Let

$$L_t = \int_0^t \frac{K_s^2 - 1}{K_s^2 + 1} ds = t - \int_0^t \frac{2 ds}{K_s^2 + 1}.$$

Note that $-t \leq L_t \leq t$ and if $p \in \mathbb{R}$,

$$\partial_t [e^{p(L_t - t)}] = e^{p(L_t - t)} \frac{-2p}{K_t^2 + 1}, \quad (65)$$

As in (62), we let

$$\sigma(t) = \int_0^t e^{2as} [K_s^2 + 1] ds \geq \int_0^t e^{2as} ds = \frac{1}{2a} [e^{2at} - 1], \quad (66)$$

Since σ is strictly increasing, we can define $\rho = \sigma^{-1}$, and the last inequality implies that there is a C' such that

$$\rho(e^{2at}) \leq t + C'.$$

We let

$$\begin{aligned} \theta &= \theta_q(r) = \frac{r}{2} + qr + \frac{r^2}{2}, \\ N_t &= e^{\theta(L_t - t)/2} e^{(\theta - \frac{r}{2})t} (K_t^2 + 1)^{r/2}. \end{aligned}$$

Although all the quantities above are defined in terms of K_t , it is useful to note that if $\theta = 2\lambda a$, then in the notation of the previous section,

$$|h'_{\sigma(t)}(z)| = e^{aL_t}, \quad Y_{\sigma(t)} = e^{at}, \quad \frac{X_{\sigma(t)}}{Y_{\sigma(t)}} = K_t,$$

$$M_{\sigma(t)} = e^{a\lambda L_t} e^{at(r - \frac{r^2}{4a})} [K_t^2 + 1]^{r/2} = N_t.$$

Since N_t is M_t sampled at an increasing family of stopping times, the next proposition is no surprise.

Proposition 5.6. *N_t is a positive martingale satisfying*

$$dN_t = N_t \frac{rK_t}{\sqrt{K_t^2 + 1}} dB_t, \quad N_0 = (x^2 + 1)^{r/2}. \quad (67)$$

In particular,

$$\mathbb{E}^x [e^{\theta L_t/2} (K_t^2 + 1)^{r/2}] = (x^2 + 1)^{r/2} e^{\zeta t/2},$$

where

$$\zeta = \zeta(r) = r - \theta = \frac{r}{2} - qr - \frac{r^2}{2}.$$

Proof. We have already noted that $N_t \geq e^{-t/2}$. Itô's formula gives

$$\begin{aligned} d\sqrt{K_t^2 + 1} &= \sqrt{K_t^2 + 1} \left[\frac{K_t^2(\frac{1}{2} - r - q) + \frac{1}{2}}{K_t^2 + 1} dt + \frac{K_t}{\sqrt{K_t^2 + 1}} dB_t \right], \\ d(K_t^2 + 1)^{r/2} &= (K_t^2 + 1)^{r/2} \left[\frac{(-qr - \frac{r^2}{2})K_t^2 + \frac{r}{2}}{K_t^2 + 1} dt + \frac{rK_t}{\sqrt{K_t^2 + 1}} dB_t \right]. \end{aligned}$$

Using this and (65), we see that N_t satisfies (67). If we use Girsanov's theorem, the weighted paths satisfy

$$dK_t = \left(\frac{1}{2} - q \right) K_t dt + \sqrt{K_t^2 + 1} dW_t, \quad (68)$$

where W_t is a standard Brownian motion in the new measure. It is straightforward to show (see Lemma 5.7) that this equation does not have explosion in finite time, and hence we can see that N_t is actually a martingale from which we conclude the final assertion. \square

We let $\tilde{\mathbb{E}}^x, \tilde{\mathbb{P}}^x$ denote probabilities with respect to the new measure, i.e., if A is an event measurable with respect to $\{B_s : s \leq t\}$, then

$$\tilde{\mathbb{P}}^x(A) = \tilde{\mathbb{E}}^x[1_A] = \mathbb{E}^x[N_t 1_A].$$

If x is omitted, then $x = 0$ is assumed. Then K_t satisfies (68) where W_t is a standard Brownian motion with respect to $\tilde{\mathbb{P}}$.

5.5 The SDE (68)

In this subsection we study the one-variable SDE (68). Note that if we do a time change to (68), we get the equation

$$d\hat{K}_t = \left(\frac{1}{2} - q \right) \frac{\hat{K}_t}{\hat{K}_t^2 + 1} dt + d\hat{W}_t, \quad (69)$$

which is very similar to the standard Bessel equation. In fact, (69) looks like a Bessel equation when K_t is large but is better behaved than the Bessel equation for K_t near zero. In analogy, we expect the behavior of (68) to have three distinct regimes: $q > 0, q = 0, q < 0$. We will consider only the $q > 0$ case for which the process is positive recurrent. We define

$$\mu = \mu_q = \frac{1 - 2q}{1 + 2q}. \quad (70)$$

Lemma 5.7. *For every $q > 0$, there is a $\delta > 0$ such that if K_t satisfies (68) and $x^2 \geq 1$,*

$$\mathbb{P}^x \left\{ \frac{x^2}{2} \leq K_s^2 \leq 2x^2, 0 \leq s \leq \delta \right\} \geq \frac{1}{2}. \quad (71)$$

In particular, the equation does not have explosion in finite time.

Proof. Straightforward and left to the reader. □

Lemma 5.8. *Suppose K_t satisfies (68) with $q > 0$. Let*

$$\phi(x) = \int_0^{|x|} (s^2 + 1)^{q-\frac{1}{2}} ds,$$

$$\rho = \rho(y) = \inf\{t : K_t^2 = 0 \text{ or } K_t^2 \geq y^2\}.$$

Then, $\phi(K_{t \wedge \rho})$ is a martingale. In particular, if $0 \leq x^2 \leq y^2$,

$$\mathbb{P}^x[K_\rho^2 = y^2] = \frac{\phi(x)}{\phi(y)}.$$

Proof. We may assume $x > 0$. Since

$$\phi''(x) = \frac{(2q-1)x}{x^2+1} \phi'(x),$$

the first assertion follows from Itô's formula. The second assertion follows from the optional sampling theorem since $\phi(0) = 0$. □

Proposition 5.9. *Suppose K_t satisfies (68) with $q > 0$.*

- *The process is positive recurrent with invariant density*

$$u_q(x) := \frac{\Gamma(q + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(q)} \frac{1}{(x^2 + 1)^{q+\frac{1}{2}}}.$$

In particular, there is a c_ such that*

$$u_q(x) \sim c_* x^{-2q-1}, \quad x \rightarrow \infty. \tag{72}$$

•

$$\int_{-\infty}^{\infty} \frac{x^2 - 1}{x^2 + 1} u_q(dx) = \mu, \tag{73}$$

where μ is as defined in (70).

- *There is a c such that for all $l \geq 0$, $b \geq 0$,*

$$\mathbb{P}\{K_t^2 + 1 \geq b^2 \text{ for some } l \leq t \leq l+1\} \leq \frac{c}{(1+b^2)^q}. \tag{74}$$

- *If $\alpha < q$, then*

$$\mathbb{E}[(K_t^2 + 1)^\alpha] \leq \int_0^\infty (x^2 + 1)^\alpha u_q(x) dx < \infty. \tag{75}$$

•

$$\mathbb{P}^x\{K_t^2 + 1 \geq r^2 \text{ for some } l \leq t \leq l+1\} \leq c \left(\frac{1+x^2}{1+r^2} \right)^q. \quad (76)$$

- There is a c_q such that for all x and all $y > |x|$, if $\rho = \rho(y) = \inf\{t : K_t^2 = 0 \text{ or } K_t^2 = y^2\}$, then

$$\mathbb{P}^x\{K_\rho^2 = y^2\} \leq c_q \left(\frac{x^2 + 1}{y^2 + 1} \right)^q. \quad (77)$$

Moreover, for all $0 < q_1 < q_2 < \infty$, c_q can be chosen uniformly over $q \in [q_1, q_2]$.

Proof. It is standard (see, e.g., [4, p. 281]) that the stationary density of the equation

$$dK_t = m(K_t) dt + \sigma(K_t) dW_t$$

is a multiple of

$$\frac{1}{\sigma^2(x)} \exp \left\{ 2 \int_0^x \frac{m(y)}{\sigma^2(y)} dy \right\}$$

(in the literature on one-dimensional diffusions a multiple of $u_q(x) dx$ is called the speed measure). The computation of u_q is then a straightforward calculation (note that $q > 0$ is needed for the density to be integrable) and

$$\int_{-\infty}^{\infty} \frac{u_q(x) dx}{x^2 + 1} = \frac{\Gamma(q + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(q)} \frac{\Gamma(\frac{1}{2}) \Gamma(q + 1)}{\Gamma(q + \frac{1}{2} + 1)} = \frac{q}{q + \frac{1}{2}}.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x^2 - 1}{x^2 + 1} u_q(dx) = 1 - \frac{2q}{q + \frac{1}{2}} = \frac{1 - 2q}{1 + 2q}.$$

(74) follows easily from (72) and (71) since the invariant measure of $[b/2, \infty)$ is bounded by $c b^{-2q}$. (77) follows from Lemma 5.8.

To prove (75), note that the integral is the expectation of $(K_t^2 + 1)^\alpha$ assuming K_0 has density u_q . If instead we choose $K_0 = 0$, a simple coupling argument shows that the expectation is decreased. The second inequality is immediate from the explicit form of u_q . \square

As before, let

$$L_t = \int_0^t \frac{K_s^2 - 1}{K_s^2 + 1} ds = t - \int_0^t \frac{2 ds}{K_s^2 + 1}.$$

The relation (73) implies that the typical value of L_t/t is μ . In fact, we expect that $L_t = \mu t + O(t^{1/2})$. Lemma 5.11 quantifies this. We use the standard technique of computing exponential moments to study the concentration of a distribution. The next lemma computes the moment.

Lemma 5.10. *Suppose K_t satisfies (68), $\delta \in (-\infty, q)$, and*

$$p = p(\delta) = \frac{1 + 2q}{4} \delta - \frac{\delta^2}{4}, \quad (78)$$

$$\theta = \theta(\delta) = 2p - \frac{\delta}{2} = q\delta - \frac{\delta^2}{4}.$$

Then

$$N_t = N_{t,r} := e^{pL_t} (K_t^2 + 1)^{\delta/2} e^{(\theta-p)t}.$$

is a martingale. In particular,

$$\mathbb{E}^x[e^{pL_t} (K_t^2 + 1)^{\delta/2}] = (x^2 + 1)^{\delta/2} \exp\left\{t\left(\frac{\delta}{2} - p\right)\right\}. \quad (79)$$

Moreover, if $0 \leq s < t$,

$$\tilde{\mathbb{E}}[e^{p(L_t - L_s)}] \leq c \exp\left\{(t - s)\left(\frac{\delta}{2} - p\right)\right\}. \quad (80)$$

Proof. Itô's formula shows that

$$\begin{aligned} d(K_t^2 + 1)^{1/2} &= (K_t^2 + 1)^{1/2} \left[\frac{K_t^2(\frac{1}{2} - q) + \frac{1}{2}}{K_t^2 + 1} dt + \frac{K_t}{\sqrt{K_t^2 + 1}} dW_t \right], \\ d(K_t^2 + 1)^{\delta/2} &= (K_t^2 + 1)^{\delta/2} \left[\frac{K_t^2(\frac{\delta^2}{2} - \delta q) + \frac{\delta}{2}}{K_t^2 + 1} dt + \frac{\delta K_t}{\sqrt{K_t^2 + 1}} dW_t \right]. \end{aligned}$$

Combining this with (65), we that

$$dN_t = N_t \frac{\delta K_t}{\sqrt{K_t^2 + 1}} dW_t,$$

provided that

$$\theta + \frac{\delta^2}{2} - q\delta = 0, \quad \frac{\delta}{2} + \theta - 2p = 0.$$

Hence for all $\delta \in \mathbb{R}$, N_t is a local martingale. If we use Girsanov on this, we see that paths weighted by N_t (using stopping times if necessary) satisfy

$$dK_t = \left(\frac{1}{2} - q + r\right) K_t dt + \sqrt{K^2 + 1} d\hat{W}_t,$$

where \hat{W}_t is a Brownian motion in the new measure. Note that this equation is of the form (68) with a change in the parameter. By Lemma 5.7 we know this equation does not have explosion in finite time and hence N_t is a martingale and (79) follows immediately. If $r < q$,

then we can apply Proposition 5.9. To derive (80), let $s < t$ and let \mathcal{F}_s denote the σ -algebra generated by $\{K_{s'} : s' \leq s\}$. The martingale property implies that

$$\mathbb{E}[e^{p(L_t - L_s)} (K_t^2 + 1)^{\delta/2} \mid \mathcal{F}_s] = \exp \left\{ (t - s) \left(\frac{\delta}{2} - p \right) \right\} (K_s^2 + 1)^{\delta/2}.$$

Therefore,

$$\mathbb{E}[e^{p(L_t - L_s)}] \leq \exp \left\{ (t - s) \left(\frac{\delta}{2} - p \right) \right\} \mathbb{E}[(K_s^2 + 1)^{r/2}] \leq c \exp \left\{ (t - s) \left(\frac{\delta}{2} - p \right) \right\}.$$

The last inequality holds by (75). □

Lemma 5.11. *There exists $c < \infty$ such that for all $0 \leq s \leq t$,*

$$\mathbb{E} \left[\exp \left\{ \frac{|L_t - \mu t|}{\sqrt{t}} \right\} \right] \leq c. \quad (81)$$

$$\mathbb{E} \left[\exp \left\{ \frac{|(L_t - L_s) - \mu(t - s)|}{\sqrt{t - s}} \right\} \right] \leq c. \quad (82)$$

In particular, for all $\alpha > 0$,

$$\mathbb{P} \left\{ |L_t - \mu t| \geq \alpha \sqrt{t} \right\} \leq c e^{-\alpha}, \quad (83)$$

$$\mathbb{P} \left\{ |(L_t - L_s) - \mu(t - s)| \geq \alpha \sqrt{t - s} \right\} \leq c e^{-\alpha}. \quad (84)$$

Proof. Recall that $|L_t| \leq t$, $|\mu| \leq 1$, and hence

$$\exp \left\{ \frac{|L_t - \mu t|}{\sqrt{t}} \right\} \leq e^{2\sqrt{t}}.$$

Using (74), we can see that there is a b such that

$$\mathbb{E} \left[\exp \left\{ \frac{|L_t - \mu t|}{\sqrt{t}} \right\} ; K_t^2 + 1 \geq e^{b\sqrt{t}} \right] \leq e^{2\sqrt{t}} \mathbb{P}\{K_t^2 + 1 \geq e^{b\sqrt{t}}\} \leq 1.$$

Therefore, to establish (81) it suffices to show that

$$\mathbb{E} \left[\exp \left\{ \frac{|L_t - \mu t|}{\sqrt{t}} \right\} ; K_t^2 + 1 \leq e^{b\sqrt{t}} \right] \leq c,$$

and to prove this it suffices to find c, c' such that

$$\mathbb{E} \left[e^{L_t/\sqrt{t}} (K_t^2 + 1)^{c'/\sqrt{t}} \right] \leq c \exp \{ \mu t^{1/2} \},$$

$$\mathbb{E} \left[e^{L_t/\sqrt{t}} (K_t^2 + 1)^{c'/\sqrt{t}} \right] \leq c \exp \{ -\mu t^{1/2} \}.$$

We now use (79) with

$$\delta_{\pm} = \pm \frac{4}{1+2q} \frac{1}{\sqrt{t}},$$

for which

$$p(\delta_{\pm}) = \pm \frac{1}{\sqrt{t}} + O(1/t),$$

$$\frac{\delta_{\pm}}{2} - p(\delta_{\pm}) = \pm \frac{1-2q}{1+2q} \frac{1}{\sqrt{t}} + O(1/t) = \pm \frac{\mu}{\sqrt{t}} + O(1/t).$$

This gives (81). The estimate (82) is done similarly starting with (80). (83) and (84) follow immediately from the Chebyshev inequality. \square

Proposition 5.12. *For every $\delta > 0$, there exists $c_* < \infty$ such that for all t*

$$\mathbb{P} \left\{ |L_s - \mu s| \leq c_* (s+2)^{\frac{1}{2}} \log(s+2) \text{ for all } 0 \leq s \leq t \right\} \geq 1 - \delta, \quad (85)$$

$$\mathbb{P} \left\{ |(L_t - L_s) - \mu(t-s)| \leq c_* (t-s+2)^{\frac{1}{2}} \log(t-s+2) \text{ for all } 0 \leq s \leq t \right\} \geq 1 - \delta. \quad (86)$$

Proof. Since $|\partial_t L_t| \leq 1$, for every nonnegative integer k and c_* sufficiently large,

$$\begin{aligned} \mathbb{P} \left\{ |L_s - \mu s| \geq c_* (s+2)^{\frac{1}{2}} \log(s+2) \text{ for some } k \leq s \leq (k+1) \right\} \\ \leq \mathbb{P} \left\{ |L_k - \mu k| \geq \frac{c_*}{2} (k+2)^{\frac{1}{2}} \log(k+2) \right\}. \end{aligned}$$

By (83), there is a c such that

$$\mathbb{P} \left\{ |L_k - \mu k| \geq \frac{c_*}{2} (k+2)^{\frac{1}{2}} \log(k+2) \right\} \leq \frac{c}{(k+2)^{c_*/2}},$$

and hence

$$\mathbb{P} \left\{ |L_s - \mu s| \geq c_* (s+2)^{\frac{1}{2}} \log(s+2) \text{ for some } 0 \leq s \leq t \right\} \leq c \sum_{j=0}^{\infty} \frac{1}{(j+2)^{c_*/2}},$$

which goes to zero as c_* goes to infinity. In particular, we can choose c_* sufficiently large so that this is less than δ . This gives the first estimate, and the second estimate is done similarly using

$$\begin{aligned} \mathbb{P} \left\{ |L_s - \mu s| \geq c_* (s+2)^{\frac{1}{2}} \log(t-s+2) \text{ for some } t-k-1 \leq s \leq t-k \right\} \\ \leq \frac{c}{(k+2)^{c_*/2}}. \end{aligned}$$

\square

Proposition 5.13. *There exists $u < \infty$ such that for every $\delta > 0$, there exists $c_*, \tilde{c}_* < \infty$ such that for all t ,*

$$\mathbb{P} \{ K_s^2 + 1 \leq c_* \min\{(s+1)^u, (t-s+1)^u\} \text{ for all } 0 \leq s \leq t \} \geq 1 - \delta, \quad (87)$$

Moreover, on this event we have for all $0 \leq s \leq t$,

$$\sigma(s) \leq \tilde{c}_* e^{2as} \min\{(s+1)^u, (t-s+1)^u\}.$$

Proof. Note that (74) implies that we can choose c, c_*, u such that

$$\mathbb{P} \{ K_s^2 + 1 \geq c_* (s+1)^u \text{ for some } k \leq s \leq (k+1) \} \leq \frac{c}{c_* (k+1)^2}.$$

$$\mathbb{P} \{ K_s^2 + 1 \geq c_* (t-s+1)^u \text{ for some } t-(k+1) \leq s \leq t-k \} \leq \frac{c}{c_* (k+1)^2}.$$

We then derive (87) as in the previous lemma. □

5.6 Upper bound for Theorem 5.1

Proposition 5.14. *Suppose*

$$0 \leq r < 6a - 2\sqrt{5a^2 - a}, \quad a \geq \frac{1}{4}, \quad (88)$$

$$0 \leq r < 2a + \frac{1}{2}, \quad a < \frac{1}{4}. \quad (89)$$

Then there exists a $c < \infty$ such that for all $x \in \mathbb{R}$,

$$\mathbb{E}[|h'_{s^2}(x+i)|^\lambda (R_{s^2} + 1)^{r/2}] \leq c (s+1)^{\frac{r^2}{4a}-r} (x^2 + 1)^{r-\frac{r^2}{8a}} \log^{r-\frac{r^2}{4a}}(x^2 + 2),$$

In particular, if $a > 1/4$, there exists a $c < \infty$, such that for all $x \in \mathbb{R}, y \geq 2$,

$$\mathbb{E}[|h'_{s^2}(x+i)|^d] \leq c (s+1)^{d-2} (x^2 + 1)^{1-\frac{1}{8a}} \log^{1-\frac{1}{4a}}(x^2 + 1).$$

The final assertion follows from the previous one by plugging in $r = 1$ which satisfies (88) for $a > \frac{1}{4}$.

Remark The upper bound in (46) is the special case $r = 1, x = 0$. This proposition is a statement about h_{s^2} at a fixed time s^2 . In the last few sections, we saw that it is easier to process under a particular time change. It is not trivial to obtain results about the original process at fixed times by considering the time changed process. In the proof we consider the fixed time e^{2at} , and define a stopping time τ , which can be viewed as a stopping time for the time-changed process, but for which we can prove $\tau \leq e^{2at}$.

Proof. It is easy to check that (88) and (89) imply

$$r < \min \left\{ 6a - 2\sqrt{5a^2 - a}, 2a + \frac{1}{2} \right\}.$$

Consider the martingale

$$M_s = M_{s,r}(x+i) = |h'_s(x+i)|^\lambda Y_s^{r-\frac{r^2}{4a}} (R_s^2 + 1)^{r/2}.$$

Recall that $K_s = R_{\sigma(s)} = e^{-as} X_{\sigma(s)}$, where as before,

$$\sigma(s) = \inf\{u : Y_u = e^{as}\}.$$

Let $\tau = \tau_t$ be the minimum of t and the smallest s such that

$$\sqrt{K_s^2 + 1} \geq \frac{e^{a(t-s)}}{(t-s+1)}.$$

(Note that s can be considered as the time in the time-changed process, and $\sigma(s)$ is the corresponding amount of time in the original process.) Note that

$$\sigma(\tau) = \int_0^\tau e^{2as} [K_s^2 + 1] ds \leq \int_0^t \frac{e^{2at}}{(t-s+1)^2} ds \leq e^{2at}.$$

For positive integer k , let $A_k = A_{k,t}$ be the event $\{t-k < \tau \leq t-k+1\}$. Since M_t is a martingale, $\tau \leq e^{2at}$, and the event A_k depends only on $M_s, 0 \leq s \leq \tau$, the optional sampling theorem gives

$$\mathbb{E}[M_{e^{2at}} 1_{A_k}] = \mathbb{E}[M_\tau 1_{A_k}].$$

Since Y_t increases with t , we know that on the event A_k ,

$$Y_{e^{2at}} \geq Y_\tau \geq e^{at} e^{-ak}.$$

$$R_\tau^2 + 1 \asymp e^{2ak} k^{-2}. \tag{90}$$

To compute $\mathbb{E}[M_\tau 1_{A_k}]$ one only needs to consider the time-changed process and hence we can use the results about that process. The Girsanov theorem, (76), and (90) imply that

$$\mathbb{E}[M_\tau 1_{A_k}] = M_0 \tilde{\mathbb{P}}(A_k) \leq c (1+x^2)^{\frac{r}{2}+q} e^{-2aqk} k^{2q},$$

where $q = 2a + \frac{1}{2} - r > 0$.

Choose k_0 such that

$$(k_0 - 1)^{-2} (e^{2a(k_0-1)} + 1) < x^2 + 1 \leq k_0^{-2} (e^{2ak_0} + 1),$$

and note that

$$k_0 \asymp \log(x^2 + 2), \quad e^{2ak_0} \asymp (x^2 + 1) \log^2(x^2 + 2). \tag{91}$$

Let

$$V = \bigcup_{k \leq k_0} A_k.$$

Then,

$$\mathbb{E}[M_{e^{2at}} 1_V] \leq \mathbb{E}[M_\tau 1_V] \leq \mathbb{E}[M_0] = (x^2 + 1)^{r/2}.$$

On the event V , $Y_{e^{2at}} \geq e^{at} e^{-ak_0}$, hence, if we write $s = e^{at}$,

$$\begin{aligned} s^{r-\frac{r^2}{4a}} \mathbb{E}[|h'_{s^2}(x+i)|^\lambda 1_V] &\leq e^{ak_0(r-\frac{r^2}{4a})} \mathbb{E}[M_{e^{2at}} 1_V] \\ &\leq e^{ak_0(r-\frac{r^2}{4a})} (x^2 + 1)^{r/2} \\ &\leq c(x^2 + 1)^{r-\frac{r^2}{8a}} \log^{r-\frac{r^2}{4a}}(x^2 + 2). \end{aligned}$$

For $k > k_0$, we use

$$\begin{aligned} s^{r-\frac{r^2}{4a}} \mathbb{E}[|h'_{s^2}(x+i)|^\lambda 1_{A_k}] &\leq c e^{(r-\frac{r^2}{4a})ak} \mathbb{E}[M_{e^{2at}} 1_{A_k}] \\ &= c e^{(r-\frac{r^2}{4a})ak} \mathbb{E}[M_\tau 1_{A_k}] \\ &\leq c(x^2 + 1)^{\frac{r}{2}+q} k^{2q} e^{(r-\frac{r^2}{4a}-2q)ak}. \end{aligned}$$

Since $r < 6a - 2\sqrt{5a^2 - a}$, we get $r - \frac{r^2}{4a} - 2q < 0$, and hence we can sum over $k > k_0$ to get

$$\begin{aligned} s^{r-\frac{r^2}{4a}} \mathbb{E}[|h'_{s^2}(x+i)|^\lambda 1_{V^c}] &\leq c(x^2 + 1)^{\frac{r}{2}+q} k_0^{2q} e^{(r-\frac{r^2}{4a}-2q)ak_0} \\ &\leq c(x^2 + 1)^{r-\frac{r^2}{8a}} \log^{r-\frac{r^2}{4a}}(x^2 + 2). \end{aligned}$$

□

5.6.1 A lemma for another paper

In [10] we will need Lemma 5.16 Since it can be proved using the ideas in the proof of Proposition 5.14, it is convenient to include it here.

Lemma 5.15. *Suppose $a > 1/4$ and*

$$M_t = M_t(z) = |h'_t(z)|^d Y_t^{2-d} (R_t^2 + 1)^{\frac{1}{2}}.$$

If

$$2a\theta \geq \max\{\delta, \delta - 4a\delta + \delta^2\}, \quad (92)$$

then

$$N_t = M_t Y_t^{-\theta} (R_t^2 + 1)^{\frac{\delta}{2}} \quad (93)$$

is a supermartingale.

Proof. We refer to the notations and calculations in Proposition 5.2. Note that

$$N_t = |h'_t(z)|^d Y_t^{\frac{a(2-\theta)}{a}-d} (R_t^2 + 1)^{\frac{1+\delta}{2}}.$$

Then in the notation of that proposition,

$$2j_X = (1 + \delta)^2 - (4a + 1)(1 + \delta) + 2a(2 - \theta) = \delta^2 + \delta - 4a\delta - 2a\theta,$$

$$2j_Y = 1 + \delta - 4a \left(1 + \frac{1}{4a}\right) + 2a(2 - \theta) = \delta - 2a\theta.$$

Using (92), we see that $j_X, j_Y \leq 0$, and using (52) this implies N_t is a supermartingale. \square

Lemma 5.16. *Suppose $a > 1/4$, $\theta \geq 0$, $0 \leq \delta + \theta < 2q = 4a - 1$ and δ, θ satisfy (92). Then there is a c such that*

$$\mathbb{E} \left[|h'_{s^2}(x + i)|^d Y_{s^2}^{2-d-\theta} (R_{s^2}^2 + 1)^{\frac{1+\delta}{2}} \right] \leq c s^{-\theta} (x^2 + 1)^{\frac{1+\delta+\theta}{2}} \log^\theta(x^2 + 2).$$

Remark The martingale property gives

$$\mathbb{E} \left[|h'_{s^2}(x + i)|^d Y_{s^2}^{2-d} (R_{s^2}^2 + 1)^{\frac{1}{2}} \right] \leq c (x^2 + 1)^{\frac{1}{2}}.$$

This proposition can be considered as a perturbation from this. As mentioned before, it is needed in [10].

Proof. From the previous lemma we know that

$$Q_s := M_s (R_s^2 + 1)^{\delta/2} Y_s^{-\theta}$$

is a supermartingale. Let A_k, k_0, τ, V be as in the proof of Proposition 5.14 and write $s = e^{at}$. The martingale M_s corresponds to $r = 1$ in Proposition 5.14. Note that on the event A_k ,

$$(R_\tau^2 + 1)^{\delta/2} Y_\tau^{-\theta} \asymp e^{a(\delta+\theta)k} k^{-\delta} s^{-\theta}.$$

Since Q is a supermartingale and $\tau \leq s^2$,

$$\mathbb{E}[Q_{s^2} 1_{A_k}] \leq \mathbb{E}[Q_\tau 1_{A_k}] \leq c e^{a(\delta+\theta)k} k^{-\delta} s^{-\theta} \mathbb{E}[M_\tau 1_{A_k}].$$

This implies

$$\begin{aligned} s^\theta \mathbb{E}[Q_{s^2} 1_V] &\leq c \max\{k^{-\delta} e^{a(\delta+\theta)k} : k = 1, \dots, k_0\} \mathbb{E}[M_\tau 1_V] \\ &\leq c(x^2 + 1)^{\frac{1}{2}} \max\{k^{-\delta} e^{a(\delta+\theta)k} : k = 1, \dots, k_0\} \\ &\leq c(x^2 + 1)^{\frac{1}{2}} k_0^{-\delta} e^{a(\delta+\theta)k_0}. \end{aligned}$$

Recalling that $k_0^{-2} e^{2ak_0} \asymp (x^2 + 1)$, we get

$$s^\theta \mathbb{E}[Q_{s^2} 1_V] \leq c(x^2 + 1)^{\frac{1+\delta+\theta}{2}} \log^\theta(x^2 + 2).$$

Also,

$$\begin{aligned}
s^\theta \mathbb{E}[Q_{s^2} 1_{V^c}] &\leq c \sum_{k > k_0} e^{a(\delta+\theta)k} k^{-\delta} \mathbb{E}[M_\tau 1_{A_k}] \\
&\leq c \sum_{k > k_0} e^{a(\delta+\theta)k} k^{-\delta} (x^2 + 1)^{q+\frac{1}{2}} e^{-2kqa} k^{2q} \\
&\leq c e^{a(\delta+\theta-2q)k_0} k_0^{2q-\delta} (x^2 + 1)^{q+\frac{1}{2}} \\
&\leq c (x^2 + 1)^{\frac{1+\delta+\theta}{2}} \log^\theta(x^2 + 2).
\end{aligned}$$

The penultimate inequality uses the fact that $\delta + \theta < 2q$. □

5.6.2 Some corollaries

The next corollaries are useful in determining the existence of the SLE curve.

Corollary 5.17. *For every $a > 1/4$, there exist C, λ, ζ such that $\lambda + \zeta > 2$ and for all x ,*

$$\mathbb{E}[|h'_{t^2}(x+i)|^\lambda] \leq c (x^2 + 1)^{2a} (t+1)^{-\zeta}.$$

Proof. If we choose ϵ sufficiently small, then $r = 1 + \epsilon$ satisfies (88), and

$$\left[r \left(1 + \frac{1}{2a} \right) - \frac{r^2}{4a} \right] + \left[r - \frac{r^2}{4a} \right] > 2.$$

□

Corollary 5.18. *For every $0 < a < 1/4$ and every $\delta > 0$, there exist C, λ, ζ such that $\lambda + \zeta > 2$ and for all x ,*

$$\mathbb{E}[|h'_{t^2}(x+i)|^\lambda] \leq c (x^2 + 1)^{a+\frac{1}{4}+\delta} (t+1)^{-\zeta}.$$

Proof. We consider $r = 2a + \frac{1}{2} - \epsilon$. □

5.7 Lower bound for Theorem 5.1

Proposition 5.19. *Let $a > 0$ and $0 < r < 2a + \frac{1}{2}$. Let*

$$\beta = \frac{\mu}{2} = \frac{1-2q}{2+4q} = \frac{r-2a}{2+4a-2r}.$$

There exist $c > 0$ and a subpower function ϕ_0 such that the following holds. Let $X_t = X_t(i), Y_t = Y_t(i)$, and let $E = E(\phi_0, t)$ be the event that for all $1 \leq s \leq t$,

$$\sqrt{s} \max \left\{ \frac{1}{\phi(s)}, \frac{1}{\phi(t/s)} \right\} \leq Y_s \leq \sqrt{2as+1},$$

$$\frac{s^\beta}{\phi(s)} \leq |h'_s(i)| \leq s^\beta \phi(s), \quad \frac{(t/s)^\beta}{\phi(t/s)} \leq \frac{|h'_t(i)|}{|h'_s(i)|} \leq (t/s)^\beta \phi(t/s),$$

$$|X_s| \leq \sqrt{s} \min \{\phi(s), \phi(t/s)\}.$$

Then, for all $t \geq 0$,

$$\mathbb{E} \left[|h'_t(i)|^d 1_E \right] \geq c t^{\frac{r^2}{4a} - r}.$$

Remark For $r = 1, a > 1/4$,

$$\beta = 2\mu = \frac{1 - 2a}{4a},$$

which agrees with our previous definition.

Remark The idea of the proof is simple. We have already shown that for the time-changed process, a certain event has positive probability with respect to the weighted measure. In this event, K_s does not get too large, and from this we can show that the amount of time to traverse the paths in the original parametrization is about what we would expect it to be.

Proof. Let $M_t = |h'_t(i)|^\lambda Y_t^{r - \frac{r^2}{ra}} (R_t^2 + 1)^{r/2}$ be the martingale. As before, let

$$\sigma(t) = \inf \{s : Y_s = e^{at}\}.$$

If V is an event depending only on $\{B_s : s \leq \sigma(t)\}$, then the Girsanov theorem tells us that $\mathbb{E}[M_{\sigma(t)} 1_V]$ is the probability of V under the appropriately weighted measure. In our case, the paths under the weighted measure satisfy the SDE from Section 5.5. We can therefore use Propositions 5.12 and 5.13 to say that there exist positive constants c_*, u and an event $V = V_t$ such that

$$\mathbb{E} [M_{\sigma(t)} 1_V] \geq \frac{1}{2},$$

and such that on V , for $0 \leq s \leq t$,

$$|L_s - \mu s| \leq c_* (s + 2)^{\frac{1}{2}} \log(s + 2),$$

$$|(L_t - L_s) - \mu(t - s)| \leq c_* (t - s + 2)^{\frac{1}{2}} \log(t - s + 2),$$

$$K_s^2 + 1 \leq c_* \min \{(s + 1)^u, (t - s + 1)^u\}.$$

Here $K_s = R_{\sigma(s)}$ and $|h'_{\sigma(s)}(i)| = e^{aL_s}$. Roughly speaking, we would like to say on the event V ,

$$|h'_{e^{2as}}(i)| \approx |h'_{\sigma(s)}(i)| \approx e^{a\mu s} = [e^{2as}]^\beta.$$

The definition of V justifies the second relation and the equality holds by definition. We need to justify the first relation.

We claim that there exist c_1, c_2 such that on the event V ,

$$c_1 [e^{2as} - 1] \leq \sigma(s) \leq c_2 e^{2as} \min \{(s + 1)^u, (t - s + 1)^u\}, \quad 0 \leq s \leq t.$$

We have already noted that the first inequality holds for all paths. To derive the second,

$$\begin{aligned}
\sigma(s) &= \int_0^s e^{2av} (K_v^2 + 1) dv \leq c_* \int_0^s e^{2ar} (v+1)^u dr \leq c_* (s+1)^u e^{2as}, \\
\sigma(s) &\leq c_* \int_0^s e^{2av} (t-r+1)^u dv \leq c_* e^{2as} (t-s+1)^u \int_0^s e^{2a(v-s)} (s-r+1)^u dv \\
&\leq c_* e^{2as} (t-s+1)^u \int_0^\infty e^{-2ay} (y+1)^u dy \\
&\leq \tilde{c}_* e^{2as} (t-s+1)^u.
\end{aligned}$$

In particular,

$$\tilde{c}_1 e^{2at} \leq \sigma(t) \leq c_2 e^{2at}.$$

By inverting, we see that there exist c_3, c_4 such that on the event V

$$c_3 \max \{ (s+1)^{-u}, (t-s+1)^{-u} \} e^s \leq Y_{e^{2as}} \leq c_4 e^s,$$

at least for $e^{as} \geq c_3 e^{2at}$. By using properties of the Loewner equation, we see that there is a c_5 such that for all $\sigma(t) \leq s \leq c_2 e^{2at}$,

$$c_5^{-1} |h'_s(i)| \leq |h'_{\sigma(t)}(i)| \leq c_5 |h'_s(i)|. \quad (94)$$

Let \tilde{V} be the intersection of the event V with the event

$$\sup \{ |B_s - B_{\sigma(t)}| \leq e^{at}, \sigma(t) \leq s \leq c_2 e^{2at} \}.$$

By (94) and standard properties of Brownian motion

$$\mathbb{E} [|h'_{c_2 e^{2at}}(i)|^\lambda 1_{\tilde{V}}] \geq c \mathbb{E} [|h'_{\sigma(t)}(i)|^\lambda 1_{\tilde{V}}] \geq c \mathbb{E} [|h'_{\sigma(t)}(i)|^\lambda 1_V].$$

One can check that the event \tilde{V} satisfies the necessary properties. We have established the result for the time $c_2 e^{2at}$ (but every time can be written this way for some t). \square

6 Upper bound

For the sake of completeness, we sketch a proof of the upper bound for the Hausdorff dimension using a version of the argument from [11]. It suffices to show that for every $r, t < \infty$ there is a c such that for every $\epsilon > 0$ and every $z \in \mathcal{R}(r) = [-r, r] \times [1/r, r]$,

$$\mathbb{P}\{\text{dist}(z, \gamma[0, t]) \leq \epsilon\} \leq c \epsilon^{2-d}. \quad (95)$$

Indeed, if this holds, then the expected number of balls of radius ϵ need to cover $\gamma[0, t] \cap \mathcal{R}(r)$ is $O(\epsilon^{-d})$, and from this it is easy to conclude that with probability one $\dim_h[\gamma[0, \infty)] \leq d$. The Koebe-(1/4) Theorem can be used to show that $\text{dist}[z, \gamma[0, t] \cup \mathbb{R}]$ is comparable to $Y_t := Y_t/|g'_t(z)|$. Hence (95) follows from the following proposition.

Proposition 6.1. *For every $z = y(x + i) \in \mathbb{H}$,*

$$\mathbb{P}\{\Upsilon_\infty \leq \epsilon\} \sim c_* G(z) \epsilon^{2-d}, \quad \epsilon \rightarrow 0+, \quad (96)$$

where

$$G(y(x + i)) = y^{d-2} (x^2 + 1)^{\frac{1}{2}-2a},$$

and

$$c_* = 2 \left[\int_0^\pi \sin^{4a} \theta d\theta \right]^{-1}.$$

Moreover, the rate of convergence is uniform on every compact $K \subset \mathbb{H}$.

Remark The estimate (96) with \approx replacing \sim was first established in [11]. The weaker estimate is sufficient for proving the upper bound on Hausdorff dimension. In the proof of the lower bound in [1], the \approx was replaced with \asymp . There is a proof of (96) in [6] which follows more closely that proof of [11] but uses asymptotics for complex hypergeometric functions and is not very intuitive. The proof below which uses Girsanov is more natural. The strong asymptotics are used in [10] to motivate the definition of the natural parametrization, so it seems useful to give a simple proof.

Proof. Consider the usual (forward) *SLE*

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

with $U_t = -B_t$. For $z \in \mathbb{H}$, let $Z_t = Z_t(z) = X_t + iY_t = g_t(z) - U_t$, $\Theta_t = \Theta_t(z) = \arg(Z_t(z))$ and note that

$$\begin{aligned} dX_t &= \frac{a X_t}{X_t^2 + Y_t^2} dt + dB_t, & \partial_t Y_t &= -\frac{a Y_t}{X_t^2 + Y_t^2}. \\ \partial_t \Upsilon_t &= -\Upsilon_t \frac{2a Y_t^2}{(X_t^2 + Y_t^2)^2}, \\ d\Theta_t &= \frac{(1 - 2a) X_t Y_t}{(X_t^2 + Y_t^2)^2} dt - \frac{Y_t}{X_t^2 + Y_t^2} dB_t. \end{aligned}$$

Combining all of this we can show that

$$M_t := \Upsilon_t^{d-2} \sin^{4a-1} \Theta_t$$

is a local martingale with $M_0 = G(z)$. We can use the Girsanov theorem (using the stopping time $\tau_\epsilon = \inf\{t : \Upsilon_t \leq \epsilon\}$) to weight the paths by the local martingale M_t . Then,

$$d\Theta_t = \frac{2a X_t Y_t}{(X_t^2 + Y_t^2)^2} dt - \frac{Y_t}{X_t^2 + Y_t^2} dW_t,$$

where W_t is a standard Brownian motion in the new measure.

Since Υ_t decreases in t , we can do a time change so that Υ_t decays deterministically. If we choose $\sigma(t)$ so that $\hat{\Upsilon}_t := \Upsilon_{\sigma(t)} = e^{-2at}$, then $\hat{\Theta}_t = \Theta_{\sigma(t)}$ satisfies

$$d\hat{\Theta}_t = 2a \cot \hat{\Theta}_t dt + d\hat{W}_t, \quad (97)$$

where \hat{W}_t is a standard Brownian motion (in the weighted measure). If $\hat{\mathbb{E}}$ denotes expectations with respect to the new measure, then

$$e^{-2at(d-2)} \mathbb{P}\{\Upsilon_\infty \leq e^{-2at}\} = \mathbb{E}[\hat{M}_t \sin^{1-4a} \hat{\Theta}_t; \Upsilon_t \leq e^{-2at}] = M_0 \hat{\mathbb{E}}[\sin^{1-4a} \hat{\Theta}_t].$$

Therefore,

$$\mathbb{P}\{\Upsilon_\infty \leq e^{-2at}\} = G(z) e^{-2at(2-d)} e(t, \arg(z)),$$

where $e(t, \theta) = \hat{\mathbb{E}}[\sin^{1-4a} \hat{\Theta}_t \mid \hat{\Theta}_0 = \theta]$ where $\hat{\Theta}_t$ satisfies (97). One can check that the invariant density for (97) is $f(\theta) = c \sin^{4a} \theta$, and hence

$$e(\infty, \theta) = \frac{c \int_0^\pi \sin \theta d\theta}{c \int_0^\pi \sin^{4a} \theta d\theta} = c_*.$$

The final statement about uniform convergence concerns the rate of convergence to the invariant distribution. We leave the simple argument to the reader. \square

7 Continuity in capacity parametrization

Here we consider solutions of

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \quad (98)$$

where $a > 0$ is fixed and U_t is continuous. As before, let $f_t = g_t^{-1}$ and for $y > 0$, let

$$v(t, y) = \int_0^y |f'_t(U_t + ir)| dr.$$

We say that t is an *accessible* time if the limit

$$\gamma(t) = \lim_{y \rightarrow 0+} f_t(U_t + iy)$$

exists. If $v(t, y) < \infty$ for some $y > 0$ then $v(t, 0+) = 0$, t is an accessible time, and

$$|\gamma(t) - f_t(U_t + iy)| \leq v(t, y). \quad (99)$$

To show that γ is continuous in the capacity parametrization (i.e., that γ is a curve) one first has to show that each t is accessible (so that γ is well defined) and then to show continuity. The strategy to show continuity is a “ 4ϵ ”-argument,

$$\begin{aligned} |\gamma(s) - \gamma(t)| &\leq |\gamma(s) - f_s(U_s + iy)| + |\gamma(t) - f_t(U_t + iy)| \\ &\quad + |f_s(U_s + iy) - f_s(U_t + iy)| + |f_s(U_t + iy) - f_t(U_t + iy)|. \end{aligned}$$

Lemma 7.1. *Suppose that g_t satisfies (98); $y > 0$; $s < t$ are accessible times with $t - s \leq y^2/(4a)$ and $\max_{s \leq r \leq t} |U_s - U_r| \leq y/4$. Then*

$$|f'_t(U_t + iy)| \leq \frac{9}{2} e^{1/4} |f'_s(U_s + iy)| \quad (100)$$

$$|f_t(U_t + iy) - f_s(U_s + iy)| \leq 8v(s, y), \quad (101)$$

$$|\gamma(t) - \gamma(s)| \leq 25 [v(s, y) + v(t, y)]. \quad (102)$$

Proof. Recall from Section 3.1 that

$$f_t(U_t + iy) = f_s(z_0),$$

where $z_0 = F_{t-s}(U_t + iy)$ and F_r is the solution to the time-reversed Loewner equation

$$\partial_r F_r(w) = \frac{a}{U_{t-r} - F_r(w)}, \quad F_0(w) = w.$$

The imaginary part increases so we get the bound $|\partial_r F_r(x + iy)| \leq a/y$ which implies $|F_{t-s}(U_t + iy) - (U_t + iy)| \leq y/4$ and hence

$$|z_0 - (U_s + iy)| \leq |z_0 - (U_t + iy)| + |U_t - U_s| \leq y/2. \quad (103)$$

Similarly, by differentiating the equation we get the bound

$$e^{-1/4} \leq |F'_{t-s}(x + iy)| \leq e^{1/4}.$$

The distortion theorem implies $|f'_s(z_0)| \geq (2/9) |f'_s(U_s + iy)|$, and hence

$$|f'_t(U_t + iy)| \geq \frac{2}{9} e^{-1/4} |f'_s(U_s + iy)|. \quad (104)$$

The distortion theorem and (103) give

$$\begin{aligned} |f_s(z_0) - f_s(U_s + iy)| &\leq 2 |f'_s(U_s + iy)| \\ &\leq 9 e^{1/4} \min\{|f'_s(U_s + iy)|, |f'_t(U_t + iy)|\} \\ &\leq 36 e^{1/4} \min\{v(s, y), v(t, y)\} \end{aligned}$$

The estimate (101) follows from the first inequality and $|f'_s(U_s + iy)| \leq 4v(s, y)$, and (102) follows from the final inequality, (99), and the estimate $36e^{1/4} + 2 < 50$. \square

Lemma 7.2. *There exists a C_0 such that if g_t satisfies (98), $R \geq 0$, $t \leq R^2/a$ and $|U_s - U_0| \leq R$, $0 \leq s \leq t$, then $\mathbb{H} \setminus H_t$ is contained in the ball of radius $C_0 R$ about the origin.*

Proof. We leave this to the reader; see [6, Section 3.4] if one wants a proof. \square

In particular, if t is an accessible point,

$$|\gamma(t) - U_0| \leq C \max \left\{ \sqrt{t}, \max_{0 \leq s \leq t} |U_s - U_0| \right\}.$$

We say that g_t is generated by a curve if every t is an accessible time and $\gamma(t)$ is a continuous function of t . The last estimate shows that γ must be right continuous at 0. Hence we get the following.

Corollary 7.3. *Suppose g_t is a solution to (98). Let*

$$v_\delta(y) = \sup_{\delta \leq t \leq 1/\delta} v(t, y) = 0.$$

Suppose that for every $\delta > 0$, $v_\delta(0+) = 0$. Then g_t is generated by a curve γ . Moreover, if $\delta \leq s \leq t \leq s + (y^2/4a) \leq 1/\delta$ and $\max_{s \leq r \leq t} |U_s - U_r| \leq y/(4a)$, $|\gamma(s) - \gamma(t)| \leq 50v_\delta(y)$.

The following is essentially a restatement of the corollaries in Section 5.6.2. We only need the lemmas for $x = 0$.

Lemma 7.4. *If $a \neq 1/4$, there exist $c < \infty, \lambda, \zeta > 0$ with $\lambda + \zeta > 2$ such that for all x, t ,*

$$\mathbb{E} [|h'_{t^2}(i)|^\lambda] \leq c(t+1)^{-\zeta}.$$

Let λ, ζ be as in the lemma and choose $\theta < 1$ with $\zeta + \theta\lambda > 2$.

$$\mathbb{P}\{|h'_{t^2}(iy)| \geq y^{-\theta}\} \leq y^{\theta\lambda} \mathbb{E} [|h'_{t^2}(iy)|^\lambda] = y^{\theta\lambda} \mathbb{E} [|h'_{t^2/y^2}(i)|^\lambda] \leq c(t+1)^{-\zeta} y^{\theta\lambda+\zeta}.$$

Corollary 7.5. *For every $\kappa \neq 8$, there exists a $\theta < 1$ such that the following holds. Let g_t denote the conformal maps of SLE_κ and $f_t = g_t^{-1}$. Let*

$$\Theta_K = \Theta_K(\theta) = \sup \left\{ y^{-\theta} |f'_t(U_t + iy)| : \frac{1}{K} \leq t \leq K, \quad 0 < y \leq 1 \right\} < \infty.$$

Then with probability one, for every $K < \infty$, $\Theta_K < \infty$. In particular, the path is generated by a curve γ , and

$$|\gamma(t) - f_t(U_t + iy)| \leq \frac{\Theta_K}{1-\theta} y^{1-\theta}, \quad \frac{1}{K} \leq t \leq K, \quad 0 < y \leq 1.$$

Moreover, if $\epsilon < (1-\theta)/2$, there is a $C = C(\omega, \theta, K)$ such that for all $s, t \leq K$,

$$|\gamma(s) - \gamma(t)| \leq C |s - t|^\epsilon.$$

Proof. It suffices to prove the result for each positive integer K and let $D_n = \{k2^{-n} : k = 1, 2, \dots\}$. Let us fix such a K . We choose r and ζ, λ as in Corollaries 5.17 and 5.18 and choose $\theta < 1$ satisfying $\zeta + \lambda\theta > 2$. Then the proposition combined with the Borel-Cantelli Lemma shows that with probability one

$$\sup \left\{ y^{-\theta} |f'_t(U_t + iy)| : y \in D_n \cap [0, 1], t \in D_{[2n(1+\epsilon)]} \cap [0, K], n = 1, 2, \dots \right\} < \infty.$$

Also, it is known that with probability one.

$$\sup \left\{ \frac{|U_t - U_s|}{|t - s|^{\frac{1}{2}-\epsilon}} : 0 \leq t, s \leq K \right\} < \infty.$$

We now use (104). □

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